PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

JANUARY 1894.

XXVII. *Long-Distance Telephony. By* Prof. PERRY, *F.R.S., assisted by* H. A. BEESTON[*].

WHEN resistance, capacity, self-induction, and leakage are taken into account, this subject is one of considerable difficulty. It is given to very few men to be able to discuss complicated mathematical formulæ without making mistakes—the proceedings of Scientific Societies possess many such mistakes detected and undetected—and consequently I instruct my students to experiment with their formulæ, using numerical values for their variables. The consideration of the most general problem of long-distance telephony, involving certain terminal conditions, has been taken up by Mr. Heaviside; but the ordinary mathematical physicist must find great difficulty in understanding the investigation (Phil. Mag. January 1887). Some of my students have recently obtained numerical results, neglecting the terminal conditions, which seem to me to be very instructive, and I think that the Tables will have a permanent value.

---

[*] Read June 23, 1893.

As a matter of fact, the line is supposed to be of infinite length, and we consider the state of a signal as it gets farther and farther away from the origin.

By comparing the current c at a section x centimetres from the origin with the current at $x+dx$, and properly disposing of the difference, we arrive at the equation :—

$$\frac{d^2c}{dx^2}=kl\frac{d^2c}{dt^2}+(kr+sl)\frac{dc}{dt}+src, \quad . \quad . \quad . \quad . \quad (1)$$

where (per unit length of conductor) k is the capacity, r the resistance, l the self-induction, and s the leakage conductivity. The solution which suits telephonic conditions is

$$c=a\epsilon^{-hx}\sin(pt-gx),$$

where

$$\sqrt{\frac{kpr}{2}}\sqrt{\sqrt{\left(1+\frac{p^2l^2}{r^2}\right)\left(1+\frac{s^2}{k^2p^2}\right)}\mp\left(\frac{pl}{r}-\frac{s}{pk}\right)}$$

gives the value of h if the minus sign is taken, and gives the value of g if the plus sign is taken, and $c=a\sin pt$ is the current at the origin. Of course $p=2\pi f$, where f is the frequency. Any number of such functions of any frequencies may exist simultaneously.

Two conditions must be satisfied in telephony. Taking the shrillest and gravest notes of the human voice as being of frequencies f and f', and taking therefore currents of these frequencies :—let X be the distance at which the ratio of the amplitudes of the shrill and grave currents is increased by $1/m$th of itself ; let Y be the distance at which one of the currents has altered in lag behind the other by $1/n$th of the periodic time of the more rapid one ; then it is easy to see that

$$\mathrm{X}=1\Big/\{m(h-h')\}\,^*,$$

$$\mathrm{Y}=2\pi\Big/\left\{np\left(\frac{g}{p}-\frac{g'}{p'}\right)\right\}.$$

---

[*] This is approximate. If m is not large, the true expression ought to be used,

$$\mathrm{X}=\log_\epsilon\left(1+\frac{1}{m}\right)\Big/(h'-h).$$

The letters with dashes indicate that p' or $2\pi f'$ must be taken instead of p. It is easy to see that X and Y become infinite if $\frac{l}{r} = \frac{k}{s}$.

I do not know what values of m and n would produce confusion of sound in the telephone. But as an exercise we have taken $m = 4$ and $n = 6$. We have also taken $p = 6000$, $p' = 600$.

For the first French Atlantic Cable the capacity and resistance were 0·43 microfarad and 2·93 ohms per nautical mile, so that

$$k = 2{\cdot}3215 \times 10^{-12} \text{ farads per centim.}$$

$$r = 1{\cdot}582 \times 10^{-5} \text{ ohms per centim.}$$

Mr. Beeston has calculated the distances X and Y, the lesser of which may be taken as the limiting distance for good telephony for various values of l and s.

Table I.—Limiting distances X in millions of centimetres for various amounts of leakage and self-induction. (One million centimetres are equivalent to about six miles.)

Values of $s \times 10^{10}$.	Values of $l \times 10^{10}$.						
	0	2·6373	26·373	79·118	131·863	184·61	263·73
0	0·983	1·054	1·963	5·169	10·178	17·500	33·839
·01	...	...	1·961				
·10	...	...	1·969				
1	...	...	1·999				
5	1·049	1·130	2·187	6·339	13·944	41 532	75·59
10	1·130	1·224	2·356	8·462	21·98	59·03	328·2
20	1·319	1·444	3·357	18·009	128·17	very large	131·65
40	1·754	1·965	6·250	390·11	67·95	18·94	8·75
70	2·527	2·935	17·671	39·73	7·727	4·161	2·573
100	3·447	4·134	72·03	10·00	0·650	2·111	1·617
150	5·314						
200	7·536						
250	10·090						

Table II.—Limiting distances Y in millions of centimetres for various amounts of leakage and self-induction.

Values of $s \times 10^{10}$.	Values of $l \times 10^{10}$.						
	0	2·6373	26·373	79·118	131·863	184·61	263·73
0	1·459	1·484	1·778	2·665	3·719	4·996	7·239
·01	...	...	1·783				
·10	...	...	1·800				
1	...	...	1·893				
5	1·889	1·927	2·424	4·142	6·674	10·472	20·29
10	2·426	2·482	3·297	6·639	14·130	28·88	116·36
20	3·752	3·874	5·796	18·700	82·59	very large	75·12
40	7·292	7·649	15·152	566·36	76·55	18·184	7·214
70	15·216	17·342	57·965	76·304	11·493	5·266	2·815
100	27·078	30·147	283·49	22·70	0·354	3·267	1·810
150	58·35						
200	106·43						
250	170·36						

It will be noticed that although X and Y are derived in different ways, by taking certain values for m and n they could be made much the same in value, and altering s or l seems to produce the same sort of effects on X and Y.

Mr. Beeston has drawn curves from the calculated numbers, but these need not be published.

If there is no self-induction, increasing the leakage increases the distance to which we can telephone. If there is no leakage, increasing the self-induction increases the distance.

When the amounts of s and l are not too great, increasing either increases the distance. These and other important facts are visible in the tables.

Without such tables as these and this method of study it would be almost impossible for the average mathematician to make anything of his mathematical results. Thus, for example, when equation (1) applies to such a function as $\sin pt$ it is just the same as

$$\frac{d^2c}{dx^2} = \left(kl - \frac{sr}{p^2}\right)\frac{d^2c}{dt^2} + (kr + sl)\frac{dc}{dt} . \quad . \quad . \quad . \quad (2)$$

Hence we see that the effect of leakage is to *diminish* the self-induction by the amount sr/kp^2, and to increase the resistance by the amount sl/k. But it is easy to see that if we diminish self-induction or increase resistance we do harm in telephony, and yet this kind of diminution through leakage does good. On going into the matter carefully, it is seen that it is the p being in the denominator of sr/kp^2 which produces the good effect. In fact, if l and s are small, taking $p=6000$, $p'=600$, we find

$$X \propto 1 \Big/ \sqrt{kr}\left\{1-4246\frac{l}{r}-\frac{s}{3800k}\right\}.$$

So that increasing s or l produces a good effect. Having found the mathematical reason we have not far to go to find the physical reason.

It is evident from the tables that if we had no leakage we could completely get rid of the evil effects of capacity by introducing self-induction. It is also evident that if we had no self-induction, we could completely get rid of the evil effects of capacity by introducing leakage. But when there is some leakage and some self-induction, we can in practice only mitigate the evil effect of capacity; for it is obvious that, although certain values of l and s give infinite distances, doubling or halving these values produces enormous diminution in distance, and such a constant of a cable as s may alter very greatly.

About fifteen years ago, with Prof. Ayrton I made many experiments on signalling through bare copper wires lying at the bottom of the water in the moat of Yedo in Japan. Here k and s were both very great. We had much less success than we expected, and we abandoned, perhaps too readily, our idea of a very cheap submarine cable.

The following tables are of general application. Let the numbers given in Table III. be divided by the value of $\sqrt{kr}$ for any cable or conductor of a telephonic line, and let them also be divided by the value of m which is considered suitable*, and they will become the limiting distances X in centimetres for that conductor, for the various values of $\frac{l}{r}$ and $\frac{s}{k}$ given.

---

* It is more correct to say that the numbers are to be multiplied by $\log_\epsilon\left(1+\frac{1}{m}\right)$.

TABLE III.

Values of s/k.	Values of $10^5 \times l/r$.						
	0	1·667	16·67	50	83·33	116·7	166·7
0	0·0267	0·0286	0·0533	0·1404	0·2765	0·4753	0·9191
215·4	0·0285	0·0307	0·0594	0·1722	0·3787	1·1281	2·053
600	……	……	……	……	……	……	∞
430·8	0·0307	0·0332	0·0640	0·2298	0·5970	1·6034	8·913
857	……	……	……	……	……	∞	
861·5	0·0358	0·0392	0·0912	0·4891	3·4732	2755·0	3·576
1200	……	……	……	……	∞		
1723	0·0476	0·0534	0·1698	10·596	1·8455	0·5145	0·2377
2000	……	……	……	∞			
3015	0·0686	0·0797	0·4799	1·0792	0·2099	0·1130	0·0699
4308	0·0936	0·1123	1·9564	0·2717	0·0176	0·0573	0·0439
6000	……	……	∞				
60000	……	∞					

TABLE IV.

Values of s/k.	Values of $10^5 \times l/r$.						
	0	1·667	16·67	50	83·33	116·7	166·7
0	0·0531	0·0540	0·0646	0·0969	0·1352	0·1813	0·2633
215·4	0·0687	0·0700	0·0881	0·1506	0·2427	0·3808	0·7379
600	……	……	……	……	……	……	∞
430·8	0·0882	0·0903	0·1199	0·2414	0·5139	1·0502	4·231
857	……	……	……	……	……	∞	
861·5	0·1364	0·1409	0·2108	0·6800	3·0033	111056	2·732
1200	……	……	……	……	∞		
1723	0·2652	0·2781	0·4377	20·596	2·7837	0·6613	0·2624
2000	……	……	……	∞			
3015	0·5533	0·6306	2·1079	2·775	0·4179	0·1915	0·1024
4308	0·9847	1·0963	0·1031	0·826	0·0129	0·1188	0·0658
6000	……	……	∞				
60000	……	∞					

Let the numbers in Table IV. be divided by the value of $\sqrt{kr}$ for any conductor of a telephonic line, and let them also be divided by the value of n which is considered suitable, and they will become the limiting distances Y in centimetres for that conductor, for the various values of $\frac{l}{r}$ and $\frac{s}{k}$ given.

Another way of putting the results in these tables is this :—Let K be the whole capacity of the line in farads, R its resistance in ohms, L its whole self-induction in secohms, S the whole leakage conductivity in mhos : L/R and S/K are the same as the l/r and s/k of the tables. For given values of these find the number in either table ; square it and divide by any chosen m^2 or n^2 ; this gives the product of the whole capacity K and the whole resistance R of the conductor.

Of course in all lines which have the same values of L/R and S/K, the product KR is constant. I must again draw attention to the fact that we have neglected the terminal conditions.

I have not hitherto said anything about the amplitude of the current ; in fact the receiving apparatus has been supposed to be infinitely delicate. It is obvious that $1/h$ is the

TABLE V.

Values of s/k.	Values of $\frac{e}{r} \times 10^5$.						
	0	1·667	16·67	50	83·3	116·7	166·7
0	·0127	·0167	·0197	·0314	·0402	·0476	·0567
215·4	·0124	·0130	·0188	·0283	·0340	·0389	·0331
430·8	·0122	·0128	·0177	·0257	·0295	·0315	·0329
861·5	·0118	·0123	·0167	·0218	·0233	·0236	·0232
1723	·0110	·0114	·0145	·0166	·0164	·0158	·0147
3015	·0099	·0102	·0121	·0124	·0115	·0106	·0096
4308	·0091	·0093	·0105	·0100	·0090	·0081	·0072
6462	·0079						
8616	·0071						
10770	·0065						

distance in which the amplitude becomes $1/\epsilon^{\text{th}}$ of its initial amount. If the numbers in Table V. be divided by $\sqrt{kr}$ for any line, they give the distances in which a current of frequency 955 per second has its amplitude *halved*.

The effect of leakage is in every case to diminish the amplitude of the current, making it necessary to have more sensitive receiving-instruments or more powerful sending-instruments.

If some of our clever mathematicians would for a while put aside the ambition to write original papers and would give us in one paper, however long, an exposition of Mr. Heaviside's views on the subject, he would confer great benefits upon the average electrician. Mr. Heaviside can discover new truths, and we all believe in his results when we understand them, but he seems unable to lower his reasoning to our mathematical levels. Since writing this paper I have tried to understand Mr. Heaviside's numerous papers on this subject, but I am sorry to say that I am not yet able to express a certain opinion as to the practical value, or want of value, of the preceding tables.

DISCUSSION.

Mr. Blakesley said that some eight years ago he discussed the subject when capacity and resistance were alone considered, and now pointed out that when self-induction and leakage were introduced, the equations were still of the same form. He also suggested how terminal conditions on lines of finite length might be easily taken into consideration.

Prof. Perry, in reply, said the introduction of self-induction and leakage rendered the calculations much more laborious, and that the terminal conditions were much more complicated than Mr. Blakesley supposed.

XXVIII. *Two Lecture-Room Experiments, by* W. B. Croft, *M.A.*[*] (Abstract.)

Mr. W. B. Croft, M.A., showed two Lecture-room Experiments. One, on "The Rings and Brushes in Crystals," was performed by a very simple apparatus in two ways. In the first a bundle of glass plates was used as polarizer, and a Nicol prism as analyzer. When a Nicol could not be conveniently obtained, a glass plate could be used as a reflecting analyzer. For a convergent system two glass card-counters were used, the crystal being placed between them. Very good results were produced by this simple apparatus.

In the second arrangement the crystal was placed on the eye-piece of a microscope (whose objective was removed), and covered by a tourmaline. On reflecting light up the tube by means of a piece of glass held at the proper angle, excellent results were obtained.

Another experiment, on "Electric Radiation in Copper Filings," was similar to those described by Dr. Dawson Turner at the Edinburgh meeting of the British Association. A battery, galvanometer, and glass tube containing copper filings were joined in series. Under ordinary circumstances no current passed, but immediately an electric spark was produced by an electric machine many feet away, the galvanometer was violently deflected, and remained so until the tube was tapped. On trying different materials, aluminium and copper seemed about equal, but iron not so good. Carbon allowed the current to pass always.

Discussion.

Prof. Minchin said the phenomena were strikingly like those exhibited by his "impulsion cells," for the moment a spark passed, even at a distance of 130 feet, they became sensitive to light. Very minute sparks were capable of producing the change, but by adding capacity to the sparking circuit the effect could be greatly modified. Replying to a question from Mr. Rimington, he said the change was due to electro-magnetic vibrations, and not to light emitted by the sparks.

---

[*] Exhibited October 27, 1893.

Mr. Blakesley enquired if lengthening the sparks produced greater effect on the copper filings.

Mr. Lucas asked if the resistance of a tube ever became infinite again if left for a long time.

In reply, Mr. Croft said the current sometimes passed before the spark actually occurred between the knobs. He had not left tubes for very long, and had not found the resistance reappear without tapping.

XXIX. *On the Separation of Three Liquids by Fractional Distillation.* *By* Professor F. R. Barrell, *M.A.*, *B.Sc.*, G. L. Thomas, *B.Sc.*, *and* Professor Sydney Young, *D.Sc.*, *F.R.S.*, *University College, Bristol**.

It is well known that the separation by fractional distillation of two substances which are miscible in all proportions and which do not—like propyl alcohol and water or formic acid and water—form mixtures of constant boiling-point, is usually a simple matter if there is a considerable difference in their boiling-points. The facility with which the separation can be effected depends, in fact, chiefly on this difference.

When, however, we have to deal with a mixture of three substances, the difficulty is greatly enhanced, and if the boiling-points are not far apart it may be almost, if not quite impossible to separate any quantity of the middle substance in a state of purity by the ordinary methods of fractional distillation.

The variation in the composition of the distillate from a mixture of two substances, the boiling-point of which rises constantly during the distillation, has been carefully investigated by F. D. Brown (Trans. Chem. Soc. 1879, p. 550; 1880, pp. 49 & 304; 1881, p. 517)*; and his results may be briefly stated as follows:—Calling the relative weights of the two liquids at any instant in the still W_1 and W_2, the relative weights at the same moment in the vapour (and therefore in the distillate at this instant coming over from the mixture W_1+W_2) x_1 and x_2, and the vapour-pressures of the pure substances at the boiling-point of the mixture, P_1 and P_2, the composition of the instantaneous distillate is given approximately by the equation

---

* Read November 10, 1893.

$$\frac{x_1}{x_2}=\frac{W_1}{W_2}\cdot\frac{P_1}{P_2},$$

but by substituting a constant, c, for the ratio $\frac{P_1}{P_2}$, thus

$$\frac{x_1}{x_2}=c\cdot\frac{W_1}{W_2},$$

a still better result is obtained.

The ratio $\frac{P_1}{P_2}$ would, of course, vary to some extent during the distillation ; the value of the constant, c, does not differ greatly from the mean value of the ratio $\frac{P_1}{P_2}$.

---

* For an account of the experimental work and of the theoretical conclusions relating to the distillation of pairs of liquid that are (*a*) non-miscible, (*b*) miscible within limits, (*c*) miscible in all proportions, the article by one of us on "Distillation" in Thorpe's 'Dictionary of Applied Chemistry' may be consulted. Pairs of liquids belonging to the first and second classes boil at a lower temperature than even the more volatile component when distilled alone, and no separation can be effected by fractional distillation. Among those in the third class there are some from which only one of the two substances can be so separated; in each case of this kind there is a particular mixture (the composition of which varies to some extent with the pressure) which distils at a constant temperature without change of composition, and it is this mixture that would be separated by fractional distillation from that one of the pure substances which is present in excess. In some cases—such as propyl alcohol and water—the mixture of constant boiling-point and composition boils at a lower temperature than either of the liquids when distilled alone; in other cases—for example, formic acid and water—this mixture has a higher boiling-point than either component.

It is only when the boiling-point of every possible mixture lies between those of the components that both liquids can be separated by fractional distillation; it is only to such cases that Brown's law is applicable, and, in the case of three liquids, it is only such mixtures that are considered in this paper.

This equation of Brown's may be written in the form

$$\frac{d\xi}{d\eta}=c\,\frac{\xi}{\eta},$$

where ξ=residue of liquid A at any instant,
" η= " B "

Taking L and M as the weights of A and B originally present, and $L+M=1$, we obtain by integration

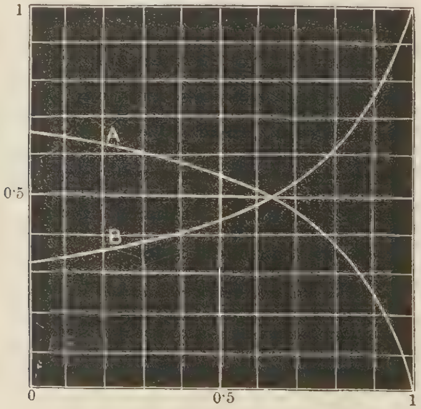
$$\frac{M}{L}y\{c+(1-c)y\}^{c-1}=c^c(1-x)^{c-1}(1-y)^c,$$

where y=quantity of the more volatile liquid A in unit weight of the distillate coming over at the instant when x is the quantity of liquid distilled.

By means of this equation we may trace the changes of composition that take place in the course of a distillation, and the variation in the composition of the distillate may be represented graphically.

To take a very simple case, let us suppose that $c=2$, and that $L=M=\frac{1}{2}$.

Fig. 1.



In the diagram (fig. 1) the amounts of distillate that have been collected are represented as abscissæ, and the relative quantities of the two liquids A and B in the distillate at any moment as ordinates.

It will be seen that the composition of the distillate alters slowly at first, then more and more rapidly, also that while the first portion of the distillate contains a considerable amount of the less volatile substance B, the last portion is very nearly free from the lower-boiling component A. These points are fully confirmed by experiment.

By fractionating a few times in the ordinary way, collecting the distillates in six or eight fractions, we shall have a very large excess of A in the first fraction, and of B in the last.

Suppose now that we have two of these fractions, one containing A and B in the ratio of 9 : 1, and the other in the ratio of 1 : 9, and that we distil these fractions separately and completely; the results will then be represented by figs. 2 and 3.

Fig. 2.

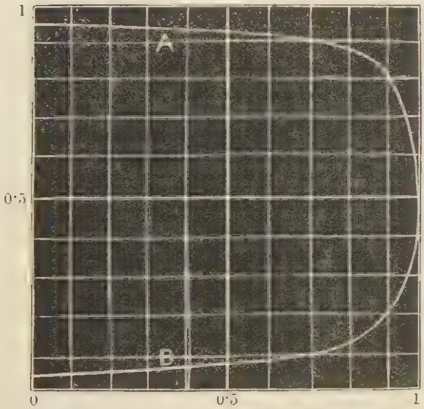
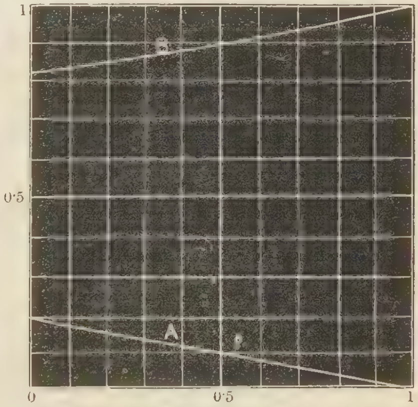


Fig 3.



In making use of this formula it is assumed that no condensation (and therefore no fractionation) goes on in the still-head, but that the vapour reaches the condenser in the same state as when first evolved from the liquid in the still. It is obvious that by using a long still-head or a dephlegmator a more rapid separation would be effected.

In any case it is evident that after a sufficient number of fractionations, the first portion of the distillate from the first fraction will be free from B, while the residue from the last fraction will be free from A.

Suppose now that our mixture contains a third substance, C, the boiling-point of which is higher than that of B.

It may be conjectured that in the progress of the distillation, the composition of the distillate at any instant will be analogous to that determined experimentally by Brown in the case of two liquids: namely, "the proportion of the three substances in the vapour forming the instantaneous distillate is the same as that of the weights of the three substances in the residue in the still, each weight being multiplied by a suitable constant which is roughly proportional to the vapour-pressure of the corresponding liquid."

With this assumption it is easy, as in the case of two liquids, to calculate curves representing graphically the progress of the distillation.

Let ξ, η, ζ = weights of the three liquids in the still at any instant;

$\therefore d\xi, d\eta, d\zeta$ = weights of the three substances in the instantaneous distillate.

By hypothesis,

$$\frac{d\xi}{a\xi} = \frac{d\eta}{b\eta} = \frac{d\zeta}{c\zeta}.$$

Integrate

$$\frac{1}{a}\log\xi = \frac{1}{b}\log\eta + k = \frac{1}{c}\log\zeta + k'.$$

Let L, M, N be the original weights of the three liquids, and $(L+M+N=1)$;

$$\therefore \frac{1}{a}\log L = \frac{1}{b}\log M + k = \frac{1}{c}\log N + k'.$$

Subtract

$$\frac{1}{a}\log\frac{\xi}{L} = \frac{1}{b}\log\frac{\eta}{M} = \frac{1}{c}\log\frac{\zeta}{N};$$

$$\therefore \left(\frac{\xi}{L}\right)^{\frac{1}{a}} = \left(\frac{\eta}{M}\right)^{\frac{1}{b}} = \left(\frac{\zeta}{N}\right)^{\frac{1}{c}}.$$

Let

$$\left.\begin{array}{l} y_1 = \text{weight of A} \\ y_2 = \quad\text{,,}\quad\ \text{B} \\ y_3 = \quad\text{,,}\quad\ \text{C} \end{array}\right\} \text{in unit weight of distillate coming over when } x = \text{weight of liquid distilled;}$$

$$\therefore \quad y_1 = \frac{d\xi}{d\xi + d\eta + d\zeta} = \frac{a\xi}{a\xi + b\eta + c\zeta};$$

$$y_2 = \frac{b\eta}{a\xi + b\eta + c\zeta}; \quad y_3 = \frac{c\zeta}{a\xi + b\eta + c\zeta}.$$

Let

$$\frac{\xi}{\mathrm{N}} = z, \quad \frac{a}{c} = \lambda + 1, \quad \frac{b}{c} = \mu + 1;$$

$$\therefore \quad \xi = \mathrm{L}z^{\lambda+1}, \quad \eta = \mathrm{M}z^{\mu+1}, \quad \zeta = \mathrm{N}z;$$

$$\therefore \quad 1 - x = \xi + \eta + \zeta = \mathrm{L}z^{\lambda+1} + \mathrm{M}z^{\mu+1} + \mathrm{N}z;$$

$$y_1 = \frac{a\mathrm{L}z^{\lambda}}{a\mathrm{L}z^{\lambda} + b\mathrm{M}z^{\mu} + c\mathrm{N}}; \quad y_2 = \frac{b\mathrm{M}z^{\mu}}{a\mathrm{L}z^{\lambda} + b\mathrm{M}z^{\mu} + c\mathrm{N}};$$

$$y_3 = \frac{c\mathrm{N}}{a\mathrm{L}z^{\lambda} + b\mathrm{M}z^{\mu} + c\mathrm{N}}.$$

The elimination of z from the expressions for x and y is impracticable, but the curves may be readily traced by treating z as an independent variable.

In the following curves (figs. 4 & 5) we have taken

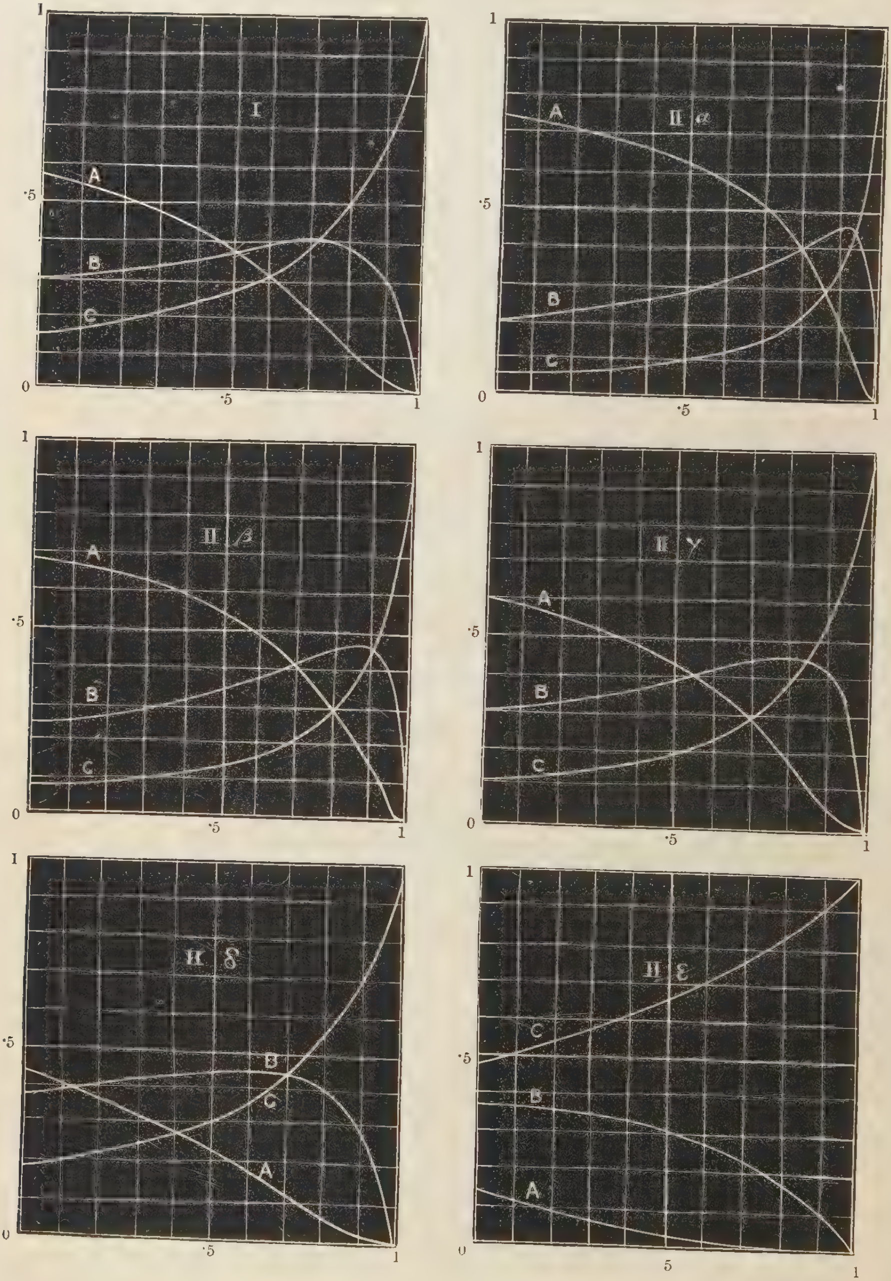
$$a=4, \quad b=2, \quad c=1, \quad [\text{and } \therefore\ \lambda=3, \quad \mu=1],$$

these being nearly proportional to the vapour-pressures of methyl, ethyl, and propyl acetates.

I. represents the first distillation; in it $\mathrm{L}=\mathrm{M}=\mathrm{N}=\frac{1}{3}$. II. α, II. β, II. γ, II. δ, II. ϵ represent what would take place if the five fractions into which I. is divided were separately distilled; the composition of these fractions being found from the curves I. to be :—

	A or L.	B or M.	C or N.
II. α	·543	·300	·157
II. β	·47	·33	·20
II. γ	·37	·365	·265
II. δ	·22	·39	·39
II. ϵ	·047	·265	·687

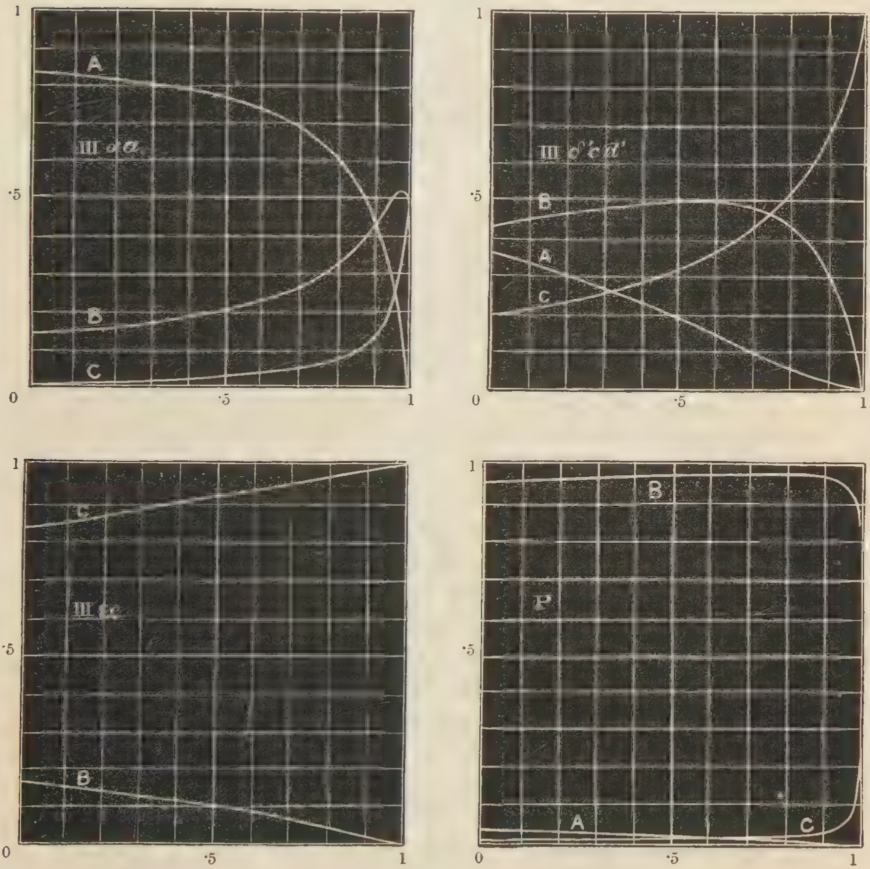
Fig. 4.


1
·5
0
·5
1
I
A
B
C
1
·5
0
·5
1
II α
A
B
C
1
·5
0
·5
1
II β
A
B
C
1
·5
0
·5
1
II γ
A
B
C
1
·5
0
·5
1
II δ
B
C
A
1
·5
0
5
1
II ε
C
B
A

It is to be noted that in II. α, which is richest in the low-boiling liquid A, the amount of A rises from ·543 to an initial value ·74, and in the first half (or at least $\frac{2}{5}$) of the distillate the aggregate amount of A rises to ·7.

In II. ε, which is richest in the high-boiling liquid C, the amount of C rises from ·687 to about ·9 in the last fifth, or ·82 in the last two-fifths of the distillate; whereas in II. δ, which is richest in B, the middle liquid, the amount of B rises merely from ·39, its initial value, to about ·45, its value

Fig. 5.



for the fifth which is collected just after the first half (·5 to ·7). It is curious that in each of these distillations B rises to practically the same maximum value, viz. ·45, the effect of the varying richness of the fractions in regard to B being merely to widen out or contract the B curve.

III. αa, III. δ "$c d$," III. ϵe represent the distillation of those fractions of II. which are richest in A, B, and C respectively, and give some indication of the processes going on at the important stages of the third fractionation. In these A rises from ·7 to ·82, B from ·45 to ·5, C from ·9 to ·98. Here we notice that, owing to the small amount of C present in III. αa (whose original composition is A ·696, B ·239, and C ·065), B attains a higher maximum (·525) than it does (·496) in III. δcd, which is far richer in B originally [A ·182, B ·44, C ·378].

P represents a distillation at an advanced stage of the fractionations—of a fraction rich in B. Its initial composition is A ·02, B ·96, C ·02. B is only advanced from ·96 to ·97 at the most favourable part of the curve, and is nowhere very free from C ; this points to the difficulty of obtaining B in a pure state by the ordinary method of fractionation.

The calculations for III. δcd are appended as a specimen of the system employed (see table opposite): with the aid of the slide-rule they are very quickly effected.

Apart from the consideration of the above hypothesis and formulæ, it is clear that after five or six fractional distillations, carried out in the ordinary manner (but with a larger number of fractions than when only two substances are present), the first fraction will contain a large excess of A, a little of B, but very little, if any, of C ; the middle fraction will contain an excess of B with moderate quantities of both A and C ; the last fraction will consist almost entirely of C, the quantity of A, if present at all, being much smaller than that of B.

Now let us suppose that these fractions are distilled separately and completely. The first portion of the distillate from the first fraction will consist of A in a nearly pure state, and, even if it contains a little of B, will be free from C. Of

$L = \cdot 182.$ $\quad 1-x = \cdot 182\,z^4 + \cdot 44\,z^2 + \cdot 378\,z\,;$

$M = \cdot 44.$

$N = \cdot 378.$ $\quad y_3 = \dfrac{378}{378 + 880\,z + 728\,z^3}.$

z.	$\cdot 378\,z$.	$\cdot 44\,z^2$.	$\cdot 182\,z^4$.	$1-x$.	$880\,z$.	$728\,z^3$.	Denominator.	y_3.	y_2.	y_1.
0	0	0	0	0	0	0	378	1	0	0
·1	·038	·004	·000	·042	88	1	467	·810	·188	·002
·2	·075	·018	·000	·093	176	6	560	·675	·314	·011
·3	·114	·040	·001	·155	264	20	662	·571	·399	·030
·4	·151	·070	·005	·226	352	46	776	·488	·453	·059
·5	·189	·110	·011	·310	440	91	909	·416	·484	·100
·6	·227	·158	·024	·409	528	157	1063	·356	·496	·148
·7	·265	·216	·044	·525	616	249	1243	·304	·495	·200
·8	·303	·282	·074	·659	704	373	1455	·260	·483	·256
·9	·340	·356	·119	·815	792	530	1700	·222	·466	·312
1·0	·378	·440	·182	1·000	880	728	1986	·190	·443	·367

the middle fraction the middle portion (or probably a rather later one) will be purest, but will contain a little of both A and C. The purest part of the last fraction will be the last portion (the residue in the still) and will be quite free from A though it may contain a little of B.

By proceeding with the fractionation in the usual manner we could obtain A and C in a pure state by separating each time the first small portion of the first distillate and the residue from the last distillate : but we should have to carry the fractionation a very great deal farther before we could obtain B in a state of purity. We should, in fact, have to reduce the quantities of A and C in this fraction until they both disappeared from the middle portion of the distillate, and an idea of the difficulty of attaining such a result may be gathered from the diagram P.

With substances whose boiling-points are not very far apart the rise of temperature during such a distillation would be almost imperceptible, and in the fractionation it would be necessary to collect a large number of fractions above and below the true boiling-point of B with very small temperature-ranges. It would be necessary also to read the temperature with extreme care, and to make very accurate corrections for changes of the barometer (which would be impossible unless the vapour-pressures of the substance had previously been determined).

It has been pointed out, however, that after the mixture has been fractionated a few times, the first portion of the first distillate is free from C, while the residue from the last distillate is free from A. If, therefore, after the fifth or sixth fractionation we remove these first and last portions each time, we shall gradually accumulate two quantities of liquid, one (the first portions) containing only A and B, the other (the residues) only B and C, and these may be separately fractionated in the ordinary way.

It is to be noticed also that as a quantity of liquid is removed each time, the total amount to be distilled, and therefore the time required for the distillation, gradually diminishes.

In arranging the fractions—as regards temperature—it is advisable, if the true boiling-points of the liquids are known, to take these temperatures and also the middle temperatures between the boiling-points of A and B and of B and C as definite points, and to arrange the fractions as far as possible symmetrically about these points. There will thus be fractions of small and diminishing temperature-range above and below the boiling-point of B—and at the boiling-points of A and C after a few fractionations and until these substances are to a large extent eliminated—and fractions of large and increasing temperature-range above and below the two middle temperatures.

As the preliminary fractionation proceeds, more and more of A and C will be eliminated, so that after a time the boiling-point of the first portion of the lowest fraction will gradually rise, while that of the residue from the last fraction will gradually fall until either A or C disappear, or, if we proceed still further, until we have nothing but B left. Much time may therefore be saved if, instead of adding all the first portions of the lowest fractions together (and the same remark applies to the residues from the last fractions), we collect them in six or more different quantities. The final fractionations will then be in a forward state when the preliminary one is completed.

We shall in this manner obtain two quantities of B, one by the final fractionation of the first portions of the distillates from the lowest fractions of the preliminary fractionations, the other by that of the residues from the preliminary fractionations, and we can test the purity of the substance by comparing the boiling-points and the specific gravities of the two samples so obtained.

In order to test the value of this method, a mixture of 200 cub. cent. of methyl acetate (B.P. 57·1), 250 cub. cent. of ethyl acetate (B.P. 77·15), and 200 cub. cent. of propyl acetate* (B.P. 101·55) was distilled from a flask provided with a still-head one metre in length. The distillate was collected in four approximately equal fractions, numbered 5, 8, 11, and 14 under F in the table (I.) below. The observed temperatures (reduced to 760 millim.), the range of temperature for each fraction (Δt), the weight of each fraction (Δw), and the ratio of the weight to the temperature-range $\left(\frac{\Delta w}{\Delta t}\right)$ are also given in the tables. The second fractionation (II.) was carried out in the following manner:—the first fraction from I. (No. 5, B. P. 63°·8 to 71°·3) was distilled and the distillate collected in a separate receiver (No. 4) until the temperature rose to 63°·8 *, when receiver No. 5 was substituted for it and the distillation continued until the temperature rose to 71°·0. The gas was then turned out, and the second fraction from I. (No. 8, B. P. 71°·3 to 77°·8) added to the residue in the flask. Heat was again applied and the distillate collected in receiver No. 5 until the temperature again rose to 71°·0, when it was replaced by receiver No. 8, and the distillation was continued until the temperature rose to 77°·1. The third fraction from I. (No. 11, B. P. 77°·8 to 89°·2) was then added to the residue in the flask, and the distillate collected in No. 8 until the temperature again rose to 77°·1, when No. 11 was put in its place. When the temperature had risen to 84°·4 a new receiver, No. 13, was substituted for No. 11, and the distillation was continued until the thermometer registered 91°·7. The last fraction from I. (No. 14, B.P. above 89°·2) was then added to the residue in the flask and the distillate collected in No. 13 until the temperature rose again to 91°·7, after which fractions were collected in No. 14 from 91°·7 to 98°·4, and No. 15 from 98°·4 to 101°·5, when the distillation was stopped, and the residue poured into receiver No. 17.

---

* The esters had previously been purified with great care, and were distilled separately from phosphorus pentoxide immediately before being mixed. Special precautions were also taken to avoid absorption of moisture during the fractionations. The methods of preparation and purification and proofs of the purity of the esters are fully described in the Trans. Chem. Soc. lxiii. p. 1194.

---

* The actual temperature was 63°·5, the barometric pressure being 751·1 millim.; the value of $\frac{dp}{dt}$ at the boiling-point is 27 millim. per degree for methyl acetate, 25 millim. for ethyl acetate, and 23 millim. for propyl acetate. All the temperatures are corrected to 760 millim. by means of these constants.

I.

F.	*t* corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w.}{\Delta t}$
5.	63°·8–71°·3	7°·5	141·4	18·9
8.	71·3–77·8	6·5	145·2	22·3
11.	77·8–89·2	11·4	146·9	12·8
14.	above 89·2	……	143·3	
			576·8	

II.

F.	*t* corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w.}{\Delta t}$
4.	60°·9– 63°·8	2°·9	59·7	20·6
5.	63·8– 71·0	7·2	134·2	18·6
8.	71·0– 77·1	6·1	98·4	16·1
11.	77·1– 84·4	7·3	67·2	9·2
13.	84·4– 91·7	7·3	51·0	7·0
14.	91·7– 98·4	6·7	86·1	12·7
15.	98·4–101·5	3·1	63·7	20·6
17.	above 101·5	……	14·3	
			574·6	

III.

F.	*t* corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w.}{\Delta t}$
3.	58°·4 – 60°·95	2°·55	36·5	14·3
4.	60·95– 63·85	2·9	77·2	26·6
5.	63·85– 71·0	7·15	100·6	14·0
8.	71·0 – 77·15	6·15	75·4	12·4
11.	77·15– 84·4	7·25	67·7	9·3
13.	84·4 – 91·65	7·25	59·6	8·2
14.	91·65– 98·4	6·75	46·7	6·9
15.	98·4 –101·45	3·05	78·3	26·1
17.	101·45–101·55	0·1	19·7	197·0
Z …	………	……	10·0	
			571·7	

IV.

F.	*t* corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w.}{\Delta t}$
2.	57°·7 – 58°·6	0°·9	17·2	19·1
3.	58·6 – 61·0	2·4	58·1	23·7
4.	61·0 – 63·9	2·9	53·1	18·3
5.	63·9 – 71·05	7·15	85·1	11·8
8.	71·05– 77·15	6·1	66·6	10·9
11.	77·15– 84·3	7·15	84·7	11·8
13.	84·3 – 91·6	7·3	40·8	5·6
14.	91·6 – 98·4	6·8	33·8	5·0
15.	98·4 –100·95	2·5	44·1	17·6
16.	100·95–101·45	0·5	38·7	77·4
17.	101·45–101·55	0·1	24·2	242·0
Z …	………	……	20·2	
			566·6	

V.

F.	*t* corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w.}{\Delta t}$
2.	57°·1 – 58°·2	1°·1	19·4	17·6
3.	58·2 – 60·7	2·5	69·9	28·0
4.	60·7 – 63·85	3·15	49·0	15·3
5.	63·85– 71·0	7·15	69·0	9·6
8.	71·0 – 77·15	6·15	71·1	11·6
11.	77·15– 83·35	6·2	85·1	13·6
13.	83·35– 90·75	7·4	33·7	4·6
14.	90·75– 98·45	7·7	33·5	4·4
15.	98·45–100·95	2·5	29·4	11·8
16.	100·95–101·45	0·5	36·7	73·4
17.	101·45–101·55	0·1	31·1	311·0
Z …	………	……	34·7	
			562·6	

VI.

F.	*t* corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w.}{\Delta t}$
1.	57°·0 – 57°·3	0°·3	6·3	21·0
2.	57·3 – 58·2	0·9	28·5	31·7
3.	58·2 – 60·1	1·9	55·0	29·0
4.	60·1 – 63·85	3·75	54·2	14·3
5.	63·85– 70·75	6·9	47·4	6·9
7.	70·75– 75·7	4·95	52·0	10·4
8.	75·7 – 77·15	1·45	33·4	22·3
11.	77·15– 78·4	1·25	34·8	29·0
12.	78·4 – 82·5	4·1	47·9	11·7
13.	82·5 – 90·2	7·7	28·1	3·6
14.	90·2 – 98·45	8·25	27·8	3·3
15.	98·45–100·95	2·5	25·6	10·2
16.	100·95–101·45	0·5	30·3	60·6
17.	101·45–101·55	0·1	34·0	340·0
Z …	………	……	50·6	
			555·9	

VII.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
1.	57°·05– 57°·25	0°·2	13·3	66·5
2.	57·25– 58·15	0·9	42·9	47·7
3.	58·15– 60·0	1·85	47·5	25·0
4.	60·0 – 63·8	3·8	43·5	11·4
5.	63·8 – 70·75	6·95	45·9	6·7
7.	70·75– 75·8	5·05	42·7	8·4
8.	75·8 – 77·2	1·4	45·0	32·1
11.	77·2 – 78·3	1·1	37·8	34·4
12.	78·3 – 82·1	3·8	41·1	10·8
13.	82·1 – 90·15	8·05	23·4	2·9
14.	90·15– 98·45	8·3	22·3	2·7
15.	98·45–100·95	2·5	17·6	7·0
16.	100·95–101 45	0·5	24·3	48·6
17.	101·45–101·55	0·1	23·7	237·0
Z ...	....	......	80·2	
			551·2	

VIII.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
A ...	° °	°	6·0	
1.	57·15– 57·35	0·2	11·7	58·5
2.	57·35– 58·25	0·9	49·1	54·6
3.	58·25– 60·15	1·9	45·4	23·9
4.	60·15– 63·8	3·65	36·0	9·5
5.	63·8 – 70·7	6·9	37·8	5·5
7.	70·7 – 75·8	5·1	41·9	8·2
8.	75·8 – 77·15	1·35	43·8	32·4
11.	77·15– 78·2	1·05	51·1	48·7
12.	78·2 – 81·1	2·9	34·1	11·8
13.	81·1 – 90·05	8·95	22·8	2·5
14.	90·05– 98·5	8·45	17·8	2·1
15.	98·5 –101·0	2·5	16·7	6·7
16.	101·0 –101·45	0·45	19·4	43·3
17.	101·45–101·55	0·1	12·6	126·0
Z ...	....	......	101·5	
			547·7	

IX.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
A ...	° °	°	11·4	
1.	57·0 – 57·25	0·25	18·6	74·4
2.	57·25– 58·0	0·75	44·5	59·3
3.	58·0 – 59·75	1·75	41·4	23·0
4.	59·75– 63·75	4·0	35·1	8·8
5.	63·75– 71·0	7·25	30·4	4·2
7.	71·0 – 75·95	4·95	39·9	8·0
8.	75·95– 77·15	1·2	47·9	39·9
11.	77·15– 78·05	0·9	57·9	64·3
12.	78·05– 80·55	2·5	27·1	10·8
13.	80·55– 89·9	9·35	20·4	2·2
14.	89·9 – 98·85	8·95	14·1	1·6
15.	98·85–101·05	2·2	12·8	5·8
16.	101·05–101·45	0·4	9·8	24·5
17.	101·45–101·55	0·1	6·3	63·0
Z ...	....	......	124·6	
			542·2	

X.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
A ...	° °	°	21·5	
1.	57·1 – 57·25	0·15	16·7	111·3
2.	57·25– 57·75	0·5	35·7	64·9
3.	57·75– 59·35	1·6	43·0	26·9
4.	59·35– 63·75	4·4	34·9	7·9
5.	63·75– 71·3	7·55	28·9	3·8
7.	71·3 – 76·1	4·8	36·9	7·7
8.	76·1 – 77·15	1·05	57·3	54·6
11.	77·15– 77·9	0·75	53·9	71·9
12.	77·9 – 79·9	2·0	23·3	11·7
13.	79·9 – 89·9	10·0	19·8	2·0
14.	89·9 – 99·3	9·4	12·5	1·3
15.	99·3 –101·1	1·8	9·7	5·4
16.	101·1 –101·5	0·4	8·4	21·0
17.	101·5 –101·55	0·05	3·5	70·0
Z ...	....	......	132·6	
			538·6	

XI.

F.	*t*. corr. to 760 mm.	Δ*t*.	Δ*w*.	$\frac{\Delta w.}{\Delta t}$
	° °	°		
A ...	....	......	31·5	
1.	57·1 – 57·2	0·1	15·2	152·0
2.	57·2 – 57·7	0·5	34·6	69·2
3.	57·7 – 59·1	1·4	36·4	26·0
4.	59·1 – 63·4	4·3	34·1	7·9
5.	63·4 – 71·5	8·1	23·9	3·0
7.	71·5 – 76 25	4·75	36·1	7·5
8.	76·25– 76·95	0·7	28·5	40·7
9.	76·95– 77·15	9·2	27·9	139·5
10.	77·15– 77·35	0·2	28·7	143·5
11.	77·35– 77·75	0 4	29·4	73 5
12.	77·75– 79·35	1·6	21·9	13·7
13.	79·35– 89·3	9·95	18·7	1·9
14.	89·3 – 99·55	10·25	10·9	1·1
15.	99·55–101·15	1·6	9·5	5·9
16.	101·15–101·45	0·3	6·0	20·0
Y ...	....	......	8·1	
Z ...	....	......	132·6	
			534·0	

XII.

F.	*t* corr. to 760 mm.	Δ*t*.	Δ*w*.	$\frac{\Delta w.}{\Delta t}$
	° °	°		
A ...	....	......	41·1	
1.	57·1 – 57·2	0·1	17·0	170 0
2.	57·2 – 57·65	0·45	30·5	67·8
3.	57·65– 58·9	1·25	30·3	24·2
4.	58·9 – 63·5	4·6	31·0	6 7
5.	63·5 – 71·9	8·4	23·8	2·8
7.	71·9 – 76·45	4·55	32·9	7·1
8.	76·45– 77·0	0·55	30·5	55·5
9.	77·0 – 77·15	0·15	27·6	184·0
10.	77·15– 77·3	0·15	36·2	241·3
11.	77·3 – 77·6	0·3	25·6	85·3
12.	77·6 – 78·85	1·25	18·7	14·9
13.	78·85– 88·9	10·05	17·9	1·8
14.	88·9 – 99·55	10·65	9·5	0·9
15.	99·55–101·15	1·6	8·1	5·1
X ...	....	......	7·8	
(Y & Z.) }	....	......	140·7	
			529·2	

XIII.

	° °	°		
A ...	....	......	50·4	
1.	57·1 – 57·2	0·1	20·4	204·0
2.	57·2 – 57·6	0·4	22·9	57·2
3.	57·6 – 58·7	1·1	26·0	23·6
4.	58·7 – 63·8	5·1	30·8	6·0
5.	63·8 – 72·35	8·55	21·3	2·5
7.	72·35– 76·65	4·3	35·2	8·2
8.	76·65– 77·05	0·4	23·2	58·0
9.	77·05-- 77·15	0·1	33·1	331·0
10.	77·15– 77·25	0·1	38·0	380·0
11.	77·25– 77·5	0·25	21·5	86·0
12.	77·5 – 78·5	1·0	17·6	17·6
13.	78·5 – 88·7	10·2	18·7	1·8
14.	88·7 – 99·7	11 0	9·3	0·8
W ...	....	......	8·1	
(X to Z.) }	....	......	148·5	
			525·0	

XIV.

	° °	°		
A ...	....	......	60·7	
1.	57·1 –57·2	0·1	14·7	147·0
2.	57·2 –57·6	0·4	23·6	59·0
3.	57·6 –58·6	1·0	22·6	22·6
4.	58·6 –64·25	5·65	29·9	5·3
5.	64·25–72·8	8·55	18·8	2·2
7.	72·8 –76 65	3·85	28·8	7·5
8.	76·65–77·05	0·4	27·1	67·7
9.	77·05–77·15	0·1	35·2	352·0
10.	77·15–77·25	0·1	43·1	431·0
11.	77·25–77·45	0·2	15·6	78·0
12.	77·45–78·2	0·75	17·4	23·2
13.	78·2 –84·7	6·5	16·0	2·4
V ...	....	......	11·4	
(W to Z.) }	....	......	156·6	
			521·5	

XV.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
A ...	....	......	70·1	
1.	57·1 –57·2	0·1	16·3	163·0
2.	57·2 –57·6	0·4	21·0	52·5
3.	57·6 –58·6	1·0	17·1	17·1
4.	58·6 –64·8	6·2	27·1	4·4
5.	64·8 –73·4	8·6	17·4	2·0
7.	73·4 –76·7	3·3	27·8	8·4
8.	76·7 –77·05	0·35	21·7	62·0
9.	77·05–77·15	0·1	43·0	430·0
10.	77·15–77·2	0·05	40·4	808·0
11.	77·2 –77·4	0·2	21·2	106·0
12.	77·4 –78·0	0·6	12·5	20·8
13.	78·0 –79·6	1·6	9·7	6·1
V ...	....	......	15·8	
(W to Z.)	....	......	156·6	
			517·7	

XVI.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
A ...	....	......	79·9	
1.	57·1 –57·2	0·1	14·6	146·0
2.	57·2 –57·6	0·4	17·3	43·2
3.	57·6 –58·6	1·0	14·6	14·6
4.	58·6 –65·35	6·75	25·0	3·7
5.	65·35–73·9	8·55	15·6	1·8
7.	73·9 –76·8	2·8	26·6	9·2
8.	76·8 –77·1	0·3	26·7	89·0
9.	77·1 –77·15	0·05	39·0	780·0
10.	77·15–77·2	0·05	39·9	798·0
11.	77·2 –77·35	0·15	24·4	162·6
12.	77·35–77·9	0·55	11·6	21·1
U ...	....	......	6·6	
(V to Z.)	....	......	172·4	
			514·2	

XVII.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
A ...	....	......	86·7	
1.	57·1 –57·2	0·1	14·4	144·0
3.	57·2 –57·6	0·4	14·3	35·7
3.	57·6 –58·6	1·0	13·5	13·5
4.	58·6 –65·8	7·2	22·4	3·1
5.	65·8 –74·4	8·6	15·9	1·8
7.	74·4 –76·85	2·45	24·7	10·1
8.	76·85–77·1	0·25	19·0	76·0
9.	77·1 –77·15	0·05	38·3	766·0
10.	77·15–77·2	0·05	48·0	960·0
11.	77·2 –77·3	0·1	22·0	220·0
12.	77·3 –77·65	0·35	9·6	27·5
U ...	....	......	10·2	
(V to Z.)	....	......	172·4	
			511·4	

XVIII.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
A ...	....	......	94·1	
1.	57·1 –57·2	0·1	11·3	113·0
2.	57·2 –57·6	0·4	13·4	33·5
3.	57·6 –58·6	1·0	10·7	10·7
4.	58·6 –66·25	7·65	21·3	2·8
5.	66·25–74·85	8·7	15·0	1·7
7.	74·85–76·85	2·0	21·4	10·7
8.	76·85–77·1	0·25	20·5	82·0
9.	77·1 –77·15	0·05	41·2	814·0
10.	77·15–77·2	0·05	49·6	992·0
11.	77·2 –77·3	0·1	21·0	210·0
T ...	....	......	6·0	
(U to Z.)	....	......	182·6	
			508·1	

XIX.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
A ...	....	......	100·3	
1.	57·1 –57·2	0·1	9·5	95·0
2.	57·2 –57·6	0·4	11·7	29·2
3.	57·6 –58·6	1·0	9·0	9·0
4.	58·6 -66·75	8·15	19·0	2·3
5.	66·75–75·35	8·6	15·7	1·8
7.	75·35–76·85	1·5	18·4	12·3
8.	76·85–77·1	0·25	18·3	73·2
9.	77·1 –77·15	0·05	39·5	790·0
10.	77·15–77·2	0·05	57·6	1152·0
11.	77·2 –77·3	0·1	13·9	139·0
T ...	....	......	9·7	
(U to Z.)	....	......	182·6	
			505·2	

XX.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
A ...	....	......	105·5	
1.	57·1 –57·15	0·05	5·9	118·0
2.	57·15–57·55	0·4	10·7	26·7
3.	57·55–58·55	1·0	8·9	8·9
4.	58·55–67·25	8·7	17·6	2·0
5.	67·25–75·95	8·7	16·6	1·9
7.	75·95–76·95	1·0	16·0	16·0
8.	76·95–77·15	0·2	13·4	67·0
9.	77·15	0·05	45·0	1888·0
10.	to 77·2		49·4	
11.	77·2 –77·25	0·05	13·0	260·0
S ...	....	......	8·6	
(T to Z.)	....	......	192·3	
			502·9	

XXI.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
A ...	....	......	105·5	
B ...	–57·15	......	11·6	
2.	57·15–57·55	0·4	7·2	18·0
3.	57·55–58·55	1·0	7·0	7·0
4.	58·55–67·2	8·65	16·3	1·8
5.	67·2 –76·3	9·1	18·2	2·0
7.	76·3 –77·0	0·7	11·2	16·0
8.	77·0 –77·15	0·15	25·3	168·6
9.	77·15 slight rise	0·0	38·7	very high
10.			35·5	
11.	77·15–77·2	0·05	14·8	296·0
S ...	....	......	16·8	
(T to Z.)	....	......	192·3	
			500·4	

XXII.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
(A & B.)	....	......	117·1	
C ...	–57·55	......	10·1	
3.	57·55–58·55	1·0	6·0	6·0
4.	58·55–67·2	8·65	12·6	1·5
5.	67·2 –76·5	9·3	19·2	2·1
7.	76·5 –77·05	0·55	15·4	28·0
8.	77·05–77·15	0·1	21·5	215·0
9.	77·15 very slight rise.	0	24·0	very high
10.			36·6	
11.	77·15	...	17·3	
R ...	....	......	9·1	
(S to Z.)	....	......	209·1	
			498·0	

XXIII.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
(A to C.)	....	......	127·2	
D ...	–58·55	......	8·6	
4.	58·55–67·2	8·65	9·1	1·1
5.	67·2 –76·7	9·4	21·7	2·3
7.	76·7 –77·1	0·4	14·4	36 0
8.	77·1 –77·15	0·05	16·6	332·0
9.	77·15	0	33·1	very high
10.			25·2	
11.	77·15		11·5	
R ...	....	......	19·7	
(S to Z.)	....	......	209·1	
			496·2	

XXIV.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
(A to C.)	....	......	127·2	
D ...	–58·55	......	10·0	
4.	58·55–67·2	8·65	6·8	0·8
5.	67·2 –75·3	8·1	15·1	1·9
6.	75·3 –76·8	1·5	7·8	5·2
7.	76·8 –77·1	0·3	12·2	40·7
8.	77·1 –77·15	0·05	14·6	292·0
9.	77·15	0	31·9	very high
10.			26·9	
Q ...	....	......	13·1	
(R to Z.)	....	......	228·8	
			494·4	

XXV.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
(A to D.)	....	......	137·2	
E ...	–67·9	......	10·2	
5.	67·9 –75·5	7·6	9·3	1·3
6.	75·5 –76·85	1·35	9·0	6·7
7.	76·85–77·1	0·25	13·2	52·8
8.	77·1 –77·15	0·05	20·2	404·0
9.	77·15	0	21·6	very high
10.			19·1	
Q ...	....	......	23·2	
(R to Z.)	....	......	228·8	
			492·3	

XXVI.

F.	t corr. to 760 mm.	Δt.	Δw.	$\frac{\Delta w}{\Delta t}$.
	° °	°		
(A to D.)	....	......	137·2	
E ...	–68·9	......	11·9	
5.	68·9 –75·7	6·8	7·1	1·0
6.	75·7 –76·9	1·2	10·4	8·7
7.	76·9 –77·1	0·2	15·4	77·0
8.	77·1 –77·15	0·05	16·4	328·0
9.	77·15	0	19·2	very high
10.			12·1	
Q ...	....	......	32·3	
(R to Z.)	....	......	228·8	
			490·8	

The third fractionation (III.) was carried out in a similar manner, a new fraction No. 3 being collected at the beginning. As the temperature rose to 101°·55, the true B.P. of propyl acetate, before the end of the last distillation, the residue in the flask consisted of this substance in a nearly pure state, and it was therefore collected in a separate flask Z, and was not redistilled during the preliminary fractionations. The residues from the subsequent fractionations up to the tenth were collected in Z, but after this, as a large amount of propyl acetate had been removed, the temperature did not reach 101°·55, and the residue from the eleventh fractionation (B. P. above 101°·45) was collected in a new flask, Y.

At the end of the twelfth fractionation the temperature rose only to 101°·15, and the residue was therefore placed in a third flask, X, and subsequently the residues were collected in W, V, &c., to Q, as shown in the tables.

It was not until the fifth fractionation that the first fraction began to boil at 57°·1, the B.P. of methyl acetate, and it was not thought advisable to separate the first portion of the first distillate until the eighth fractionation. This portion and also the corresponding ones up to the twentieth fractionation were collected together in flask A, after which the first portions were collected in B, C, D, and E.

The ratios $\frac{\Delta w}{\Delta t}$ give an indication of the purity of the various distillates, for with a pure substance, since $\Delta t = 0$, $\frac{\Delta w}{\Delta t} = \infty$, while with a very impure substance there is a rapid rise of temperature during distillation, and $\frac{\Delta w}{\Delta t}$ is consequently small.

It will be seen that the ratio for the highest fraction rises rapidly, being 197 in the third fractionation and rising to 340 in the sixth, after which it diminishes again owing to removal of propyl acetate in the residues. The value for the lowest fraction increases much more slowly, reaching a maximum of 204 in the thirteenth fractionation, and afterwards diminishing owing to removal of methyl acetate.

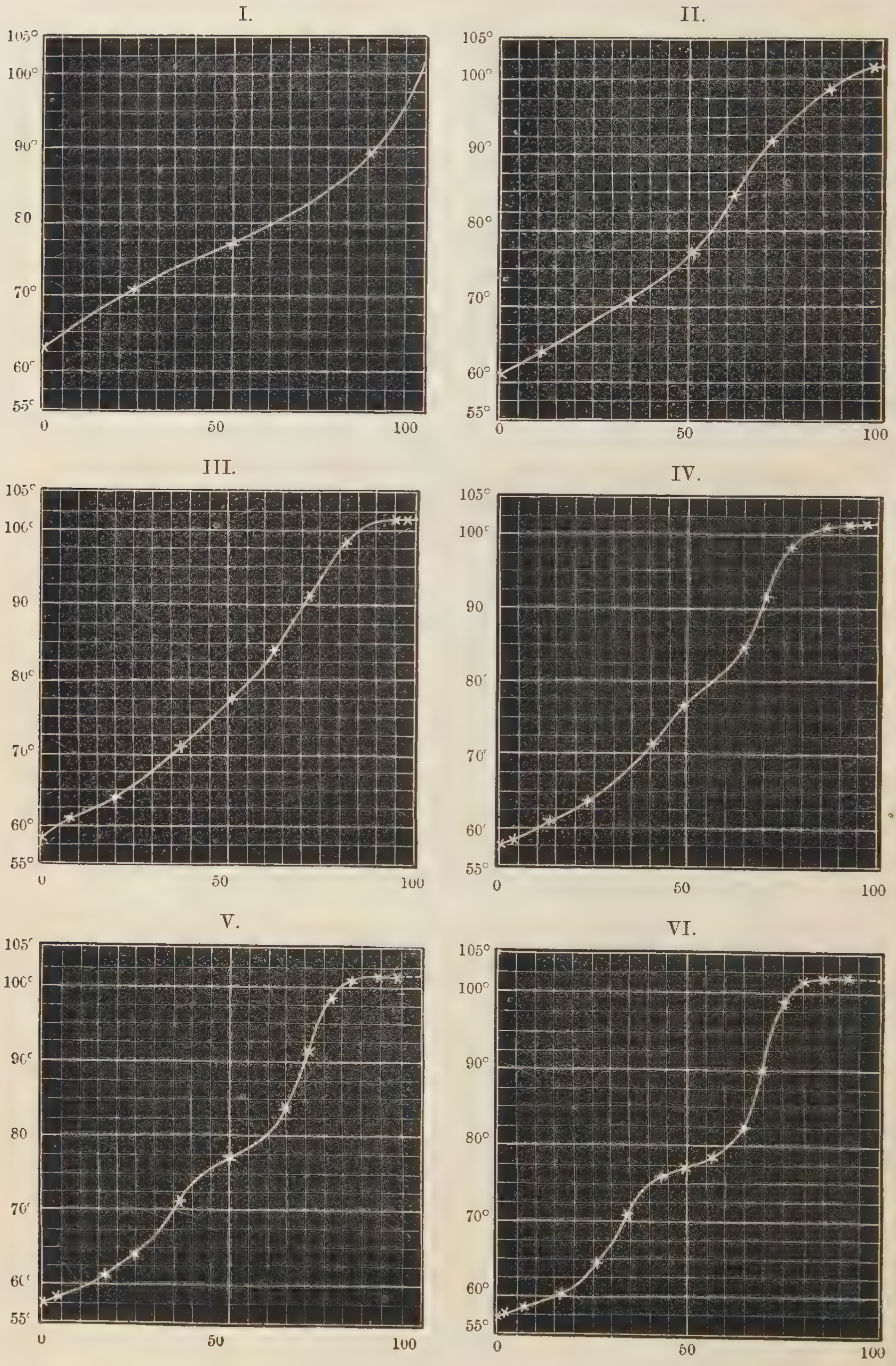
The presence of the middle substance, ethyl acetate, is not clearly shown until the fourth fractionation, when the value of $\frac{\Delta w}{\Delta t}$ for the sixth distillate (No. 9) is higher than for those above and below it; but the gradual accumulation of the ethyl acetate in the middle fractions in subsequent fractionations is clearly shown by the rise in the values of $\frac{\Delta w}{\Delta t}$ for fractions 8 and 11, and after the tenth fractionation in fractions 9 and 10. The range of temperature for fractions 9 and 10 was gradually diminished from 0°·2 each in the eleventh fractionation until no rise could be detected; there was indeed no perceptible rise of temperature during the collection of No. 11 in the twenty-second and twenty-third fractionations. It was therefore certain that after the twenty-sixth fractionation the tenth fraction was free from propyl acetate, and that the remaining fractions Nos. 5 to 10 contained only methyl and ethyl acetates.

The preliminary series of fractionations was therefore completed, no fraction now containing more than two substances.

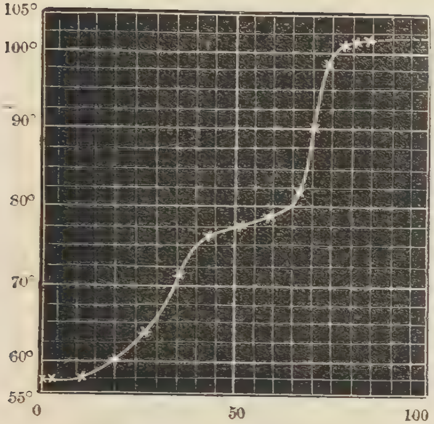
The progress of the separation is well seen by mapping the temperatures against the total weight of distillate collected. The curves I. to XII. represent the first twelve fractionations, the weight of distillate being given in each case as percentages. The dotted lines at the extremities of the later curves represent the methyl and propyl acetates removed in the first portions of the first distillates and in the residues.

The presence of the middle substance is clearly indicated in the fourth curve, but not in the earlier ones.

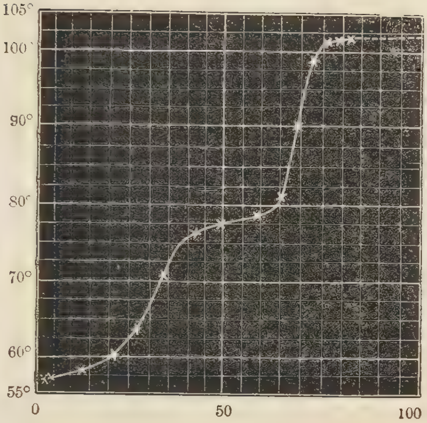
Fig. 6.


I. II. III. IV. V. VI.

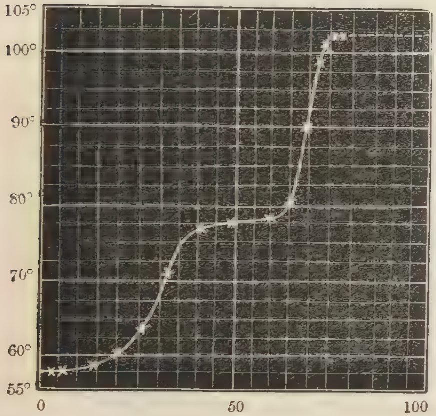
VII.



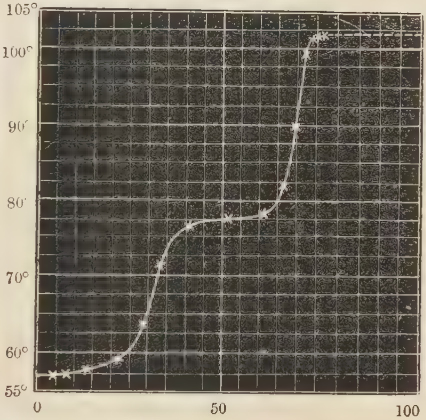
VIII.



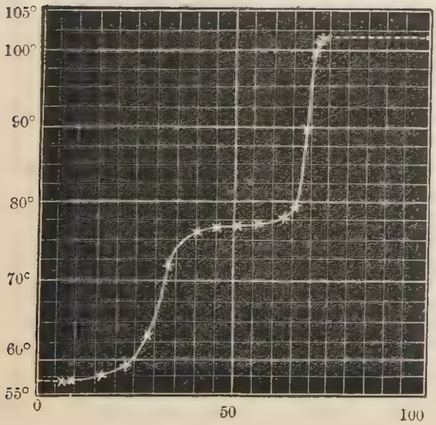
IX.



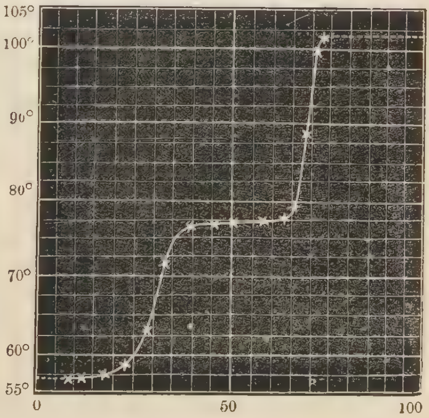
X.



XI.



XII.



Final Fractionations.

The fractions into which the esters had been separated at the end of the 26th preliminary fractionation are given below :—

Methyl and Ethyl Acetates.			Ethyl and Propyl Acetates.		
Fraction.	Temperature range.	Weight.	Fraction.	Temperature range.	Weight.
A ...	57°·1	105·5	Q ...	77°·15	32·3
B ...	below 57·15	11·6	R ...	above 77·15	19·7
C ...	„ 57·55	10·1	S......	„ 77·2	16·8
D ...	„ 58 55	10·0	T......	„ 77·3	9·7
E ...	„ 68·9	11·9	U ...	„ 77·65	10·2
5 ...	68·9–75·7	7·1	V ...	„ 79·6	15·8
6 ...	75·7–76·9	10·4	W ...	„ 99·7	8·1
7 ...	76·9–77·1	15·4	X ...	„ 101·15	7·8
8 ...	77·1–77·15	16·4	Y ...	„ 101·45	8·1
9 ...	77·15	19·2	Z ...	„ 101·55	132·6
10 ...	77·15	12·1			
		229·7			261·1

The total weight was 490·8, so that 86 grams had been lost by evaporation and through adherence to the flasks and to the funnel through which the liquid was poured into the still.

The final separation of the esters was carried out in the following manner :—

Ethyl Acetate (1).—The fractions 5 to 10 consisted of ethyl acetate with some methyl acetate. The final fractionations in this case were a continuation of the preliminary ones and were carried out in the same manner. No. 10 was almost if not quite pure ethyl acetate, and was only distilled once more; the first portion was collected in No. 9, and the residue was taken as pure. In the remaining six fractionations the residue, after the distillation of No. 9, was each time taken as pure, and the total amount of ethyl acetate thus obtained was 56·1 grams.

Ethyl Acetate (2).—The flasks Q to U contained ethyl acetate with some propyl acetate. In the six fractionations, which were carried out in the usual way, the first portion of the distillate from Q was each time taken as pure, the small residue in the still being rejected. The weight of pure ethyl acetate obtained from this series was 62·7, and the total weight was therefore 118·8 grams, or about 53 per cent. of that originally taken.

Methyl Acetate.—The flasks A to E contained methyl acetate with some ethyl acetate. Although special care had been taken to prevent absorption of moisture, it was found that a little water was present: in two fractionations, therefore, phosphorus pentoxide was placed in the distillation-bulb, which was heated by a water-bath. After the first fractionation the first portion of the distillate from the first fraction was taken as pure, the small residue in the still being each time rejected. There were altogether nine fractionations, and the amount of pure methyl acetate obtained was 88 grams, or about 48 per cent. of that taken. (The loss of this ester by evaporation would be greater than that of either of the other two.)

Propyl Acetate.—The flasks V to Z contained propyl acetate with some ethyl acetate. It was found that the first two residues were slightly acid, owing to hydrolysis of the esters by the trace of water present. These two residues were therefore collected separately and were shaken with a strong solution of potassium carbonate; the ester was then washed with water, dried with phosphorus pentoxide, and again distilled. Five fractionations were found sufficient to purify the propyl acetate, the residues being each time taken as pure (except the first two, which were treated as described). The total weight of pure propyl acetate was 126·8 grams, or 72 per cent. of that taken. (The loss by evaporation would be smallest in this case.)

The esters were then distilled from phosphorus pentoxide, and their specific gravities at 0° determined by Sprengel's method as modified by Perkin. The completeness of the separation was proved by the close agreement of the results with those obtained before mixing, as well as by the identity of the boiling-points with those previously determined.

The specific gravities are given below:—

Specific Gravities at 0°.

	Previous Determinations.	After Fractionation.
Methyl acetate * (*a*).........	·95929	·95937
„ „ (*b*).........	·95934	
Ethyl acetate * (*a*)	·92434	(1) ·92438
„ „ (*b*)	·92438	(2) ·92437
Propyl acetate * (*a*).........	·91017	·91008
„ „ (*b*).........	·91015	

* Two different samples of each ester had been purified and their specific gravities determined. The two results are given in each case, in order to show what error of experiment might be expected. The two samples of each ester were afterwards mixed together.

Apart from the comparison of the specific gravity of the ethyl acetate before mixing and after fractionation, a proof of the purity of this substance is afforded by the agreement between the specific gravities of the two samples, since, if the separation had been incomplete, one would have been contaminated with methyl acetate, the other with propyl acetate.

Discussion.

Prof. Ramsay said the paper was a most valuable one, and would be a great aid to chemists. Distillations were usually carried out by "the rule of thumb," with the result that absolutely pure liquids could rarely be obtained.

The President inquired whether the curves representing the progress of distillation could be constructed from the very complete experiments made, and so test the assumed law.

Prof. Young thought this not possible from the numbers obtained. To test the law in this way would be very laborious.

XXX. *Note on the Generalizations of* Van der Waals *regarding "Corresponding" Temperatures, Pressures, and Volumes.* *By* SYDNEY YOUNG, *D.Sc., F.R.S., Professor of Chemistry, University College, Bristol**.

IN a paper read before the Physical Society in November 1891, and published under the above title in the Philosophical Magazine for February 1892 (vol. xxxiii. p. 153; Proc. Phys. Soc. vol. xi. p. 233), the absolute temperatures and the molecular volumes (as liquid and saturated vapour) of eleven compounds were compared with those of fluorbenzene at a series of corresponding pressures.

It was pointed out, however (p. 155), that a better mode of procedure in many respects would have been to give the temperatures, pressures, and volumes of each of the twelve substances in terms of its critical constants; but as the critical volumes of several of the compounds had not been directly determined, and as none of them were known with accuracy, the method had to be abandoned.

Since then it has been shown by M. Mathias that the critical densities—and therefore the volumes—may be determined with great accuracy by the method of Cailletet and Mathias, and as the values calculated by M. Mathias differed very slightly from those obtained independently by the same method by myself, I accepted these values as correct (Phil.

Substance.	Critical Molecular Volume.		Substance.	Critical Molecular Volume.	
	Old.	Corrected.		Old.	Corrected.
Fluorbenzene........ ...	270·4	270·5	Stannic chloride .	349·3	350·4
Chlorobenzene	306·5	305·7	Ether	280·7	281·4
Bromobenzene	322·4	321·4	Acetic acid.........	170·3	170·5
Iodobenzene	348·4	347·9	Methyl alcohol ...	118·0	118·0
Benzene	256·3	256·3	Ethyl alcohol ...	166·9	166·9
Carbon tetrachloride .	276·1	275·6	Propyl alcohol ...	217·6	217·6

---

* Read November 10, 1893.

Mag. Dec. 1892). It appears, however, that there were a few small errors in the calculations, and I have therefore adopted my own values in the tables that follow. The new critical molecular volumes and those previously accepted as correct are given above ; it will be seen that the alterations are very small.

The vapour-pressures, molecular volumes, and critical constants of ten esters [methyl formate, acetate, propionate, butyrate, and isobutyrate ; ethyl formate, acetate, and propionate ; propyl formate and acetate] have recently been determined by Mr. G. L. Thomas and myself, and the absolute temperatures, pressures, and volumes of each substance have been given in terms of its critical constants (Trans. Chem. Soc. lxiii. p. 1191).

In order to allow of a comparison being made, pressures "corresponding" with the previous ones (with a few omissions) were taken, and I now propose to give the absolute temperatures and volumes of the substances previously studied in terms of their critical constants. As the data for the esters are published in full in the Trans. Chem. Soc., it is unnecessary to reproduce them here, and it will be sufficient to give the extreme values for the ten esters in each case.

It was pointed out in the earlier paper on this subject (Phil. Mag. xxxiii. p. 172) that the compounds investigated might be arranged in groups, and this is confirmed by the method of comparison now adopted. The twenty-two compounds appear to fall into four chief groups :—

I. Benzene and its halogen derivatives, carbon tetrachloride, stannic chloride, ether.
II. The three alcohols (methyl alcohol differing, however, considerably from the others).
III. Acetic acid.
IV. The ten esters.

Ratios of Absolute Temperatures at Corresponding Pressures to Absolute Critical Temperatures.—The ratios for the compounds in groups I. are the lowest ; those for the esters are mostly higher; acetic acid comes next in order; whilst the values for the alcohols are much higher than for any of the other substances.

Ratios of Boiling-Points (Abs. temps.) at Corresponding Pressures to Absolute Critical Temperatures.

Pressures C_6H_5F.	Fraction of Critical Pressure.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$	CH_3OH.	C_2H_5OH	C_3H_7OH	CH_3COOH	Ten Esters.
20	·00059	·4866	·4870	·4870	·4872	·4795	·4748	·4914	...	·5374	·5536	·5579	·5164	·4976 to ·5172
50	·00147	·5170	·5172	·5169	·5170	·5098	·5053	·5216	...	·5685	·5836	·5861	·5478	·5189 ,, ·5468
100	·00295	·5431	·5429	·5427	·5427	·5359	·5313	·5471	...	·5943	·6085	·6101	·5743	·5447 ,, ·5718
200	·00590	·5723	·5723	·5717	·5717	·5648	·5608	·5763	·5777	·6229	·6361	·6365	·6041	·5740 ,, ·5999
400	·01180	·6054	·6052	·6049	·6049	·5989	·5943	·6089	·6097	·6550	·6665	·6656	·6372	·6070 ,, ·6314
760	·02241	·6400	·6402	·6403	·6401	·6334	·6293	·6435	·6444	·6882	·6981	·6960	·6716	·6413 ,, ·6650
1500	·04423	·6827	·6833	·6833	·6830	·6765	·6728	·6855	·6866	·7277	·7355	·7326	·7133	·6838 ,, ·7059
3000	·08846	·7334	·7345	·7343	·7337	·7282	·7251	·7357	·7371	·7734	·7794	·7736	·7624	·7348 ,, ·7541
5000	·14740	·7771	·7786	·7777	...	·7725	·7697	·7788	·7798	·8119	·8161	·8118	·8031	·7778 ,, ·7949
7000	·20640	·8092	·8103	·8090	...	·8052	·8025	·8106	·8113	·8396	·8429	·8384	·8322	·8101 ,, ·8248
10000	·29490	·8464	·8466	...	...	·8429	·8411	·8463	·8472	·8708	·8738	·8690	·8651	·8452 ,, ·8595
15000	·44230	·8930	·8926	...	...	·8906	·8889	·8934	·8923	·9098	·9119	·9082	·9057	·8915 ,, ·9028
20000	·58990	·9288	·9285	...	...	·9270	·9260	·9287	·9277	·9381	·9412	·9383	·9368	·9279 ,, ·9357
25000	·73720	·9579	·9576	...	...	·9566	·9565	·9577	·9573	·9643	·9652	·9637	·9626	·9574 ,, 9624
28000	·82570	·9731	·9727	...	...	·9725	·9725	·9732	·9728	·9775	·9775	·9771	·9740	·9729 ,, ·9762
30000	·88460	·9829	·9826	...	...	·9824	·9822	·9829	·9824	·9855	·9856	·9857	·9844	·9827 ,, ·9847
32000	·94360	·9919	·9916	...	...	·9915	·9917	·9916	·9915	·9929	·9931	·9934	·9926	·9918 ,, ·9928
33912	1·00000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000 ,, 1·0000

Ratios of Volumes of Liquid at Corresponding Pressures to Critical Volumes.

Pressures C_6H_5F.	Fraction of Critical Pressure.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$	CH_3OH.	C_2H_5OH.	C_3H_7OH	CH_3COOH	Ten Esters.
20	·00059	·3381	·3366	·3360	·3357	·3359	...	·3314	...	...	·3457	·3446	·3393	·3292 to ·3324
50	·00147	·3449	·3431	·3424	·3421	·3428	·3444	·3387	...	·3413	·3515	·3501	·3460	·3358 ,, ·3392
100	·00295	·3508	·3488	·3482	·3480	·3489	·3505	·3447	...	·3455	·3565	·3552	·3523	·3419 ,, ·3451
200	·00590	·3578	·3557	·3551	·3550	·3560	·3574	·3519	...	·3522	·3626	·3610	·3597	·3490 ,, ·3523
400	·01180	·3661	·3641	·3635	·3635	·3648	·3660	·3611	·3631	·3596	·3698	·3679	·3686	·3576 ,, ·3608
760	·02241	·3755	·3737	·3732	·3732	·3742	·3756	·3705	·3729	·3680	·3781	·3759	·3788	·3671 ,, ·3704
1500	·04423	·3885	·3867	·3862	·3863	·3870	·3889	·3838	·3859	·3793	·3893	·3870	·3924	·3807 ,, ·3836
3000	·08846	·4067	·4046	·4041	·4042	·4053	·4072	·4021	·4033	·3949	·4047	·4028	·4106	.3992 ,, ·4021
5000	·14740	·4250	·4228	·4222	...	·4233	·4251	·4199	·4209	·4104	·4223	·4193	·4284	·4176 ,, ·4205
7000	·20640	·4404	·4386	·4380	...	·4389	·4402	·4356	·4368	·4243	·4367	·4340	·4439	·4334 ,, ·4364
10000	·29490	·4622	·4603	...	...	·4602	·4623	·4575	·4596	·4438	·4563	·4550	·4655	·4552 ,, ·4583
15000	·44230	·4977	...	...	...	·4960	·4983	·4948	·4948	·4760	·4891	·4897	·5003	·4912 ,, ·4940
20000	·58990	·5363	...	...	...	·5353	·5377	·5346	·5324	·5131	·5286	·5280	·5387	·5305 ,, ·5354
25000	·73720	·5855	...	...	...	·5845	·5874	...	·5833	·5601	·5772	·5775	·5871	·5800 ,, ·5855
28000	·82570	·6260	...	...	...	·6250	·6297	...	·6261	·6010	·6147	·6177	·6302	·6200 ,, ·6254
30000	·88460	·6631	...	...	...	·6613	·6654	...	·6628	·6359	·6537	·6550	·6651	·6559 ,, ·6633
32000	·94360	·7134	...	...	...	·7113	·7121	...	·7171	·6906	·7070	·7077	·7144	·7099 ,, ·7204
33912	1·00000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000 ,, 1·0000

Ratios of Volumes of Saturated Vapour at Corresponding Pressures to Critical Volumes.

Pressures C_6H_5F.	Fraction of Critical Pressure.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH	Ten Esters.
20	·00059	...	...	...	...	...	...	...	...	...	...	...	2290	... ...
50	·00147	...	...	...	...	...	...	...	...	...	...	...	1010	... ...
100	·00295	...	...	...	...	...	...	...	...	891	825	...	549	... ...
200	·00590	...	...	...	...	...	...	...	...	463	429	...	298	... ...
400	·01180	...	...	...	...	...	...	...	192	239	223	221	127	... ...
760	·02241	...	...	...	...	...	...	...	105	130	121	119	90·3	... ...
1500	·04423	55·5	55·3	54·8	55·1	54·4	52·8	54·6	55·2	67·3	62·9	61·6	48·5	56·9 to 59·4
3000	·08846	28·4	28·65	28·4	28·45	28·2	27·4	28·1	28·3	34·2	32·1	31·1	25·5	29·2 ,, 30·3
5000	·14740	17·15	17·35	17·25	...	17·25	16·75	16·95	17·1	20·4	19·3	18·6	15·7	17·4 ,, 18·45
7000	·20640	12·2	12·35	12·30	...	12·35	11·95	12·1	12·15	14·4	13·65	13·15	11·3	12·5 ,, 13·1
10000	·29490	8·37	8·43	...	...	8·49	8·19	8·41	8·38	9·82	9·33	9·01	7·91	8·56 ,, 8·92
15000	·44230	5·35	...	...	...	5·35	5·23	5·25	5·36	6·17	5·76	5·52	5·16	5·41 ,, 5·57
20000	·58990	3·73	...	...	...	3·72	3·67	3·64	3·74	4·23	3·90	3·84	3·68	3·73 ,, 3·85
25000	·73720	2·71	...	...	...	2·69	2·66	...	2·70	2·98	2·80	2·73	2·68	2·69 ,, 2·78
28000	·82570	2·22	...	...	...	2·20	2·17	...	2·21	2·40	2·31	2·25	2·20	2·21 ,, 2·28
30000	·88460	1·91	...	...	...	1·91	1·88	...	1·93	2·08	1·99	1·96	1·91	1·91 ,, 1·95
32000	·94360	1·63	...	...	...	...	1·62	...	1·61	1·76	1·65	1·69	1·62	1·61 ,, 1·64
33912	1·00000	1·00	1·00	1·00	1·00	1·00	1·00	...	1·00	1·00	1·00	1·00	1·00	1·00 ,, 1·00

In the first group the differences are not great, but the ratios for benzene and carbon tetrachloride are below, and those for stannic chloride and ether at low pressures are somewhat above the average. The influence of molecular weight and constitution on the ratios for the esters is fully discussed in the paper by Mr. Thomas and myself (Trans. Chem. Soc. lxiii. p. 1252), and it will be sufficient here to mention that there is a marked rise as the molecular weight increases. This is not the case, however, with the ratios of the volumes of liquid or of saturated vapour to the critical volumes. All the ratios appear to depend to some extent on the constitution of the esters.

Ratios of Volumes of Liquid at Corresponding Pressures to Critical Volumes.—The differences in this case are small; in general the ratios for acetic acid are the highest, and those for the esters the lowest. At the lowest pressures the values for the alcohols are higher than for the esters, but at high pressures they are lower than for any of the other substances, and this is especially the case with methyl alcohol.

Ratios of Volumes of Saturated Vapour at Corresponding Pressures to Critical Volumes.—The grouping of the compounds is well seen in this case, the differences being naturally most marked at low pressures. Acetic acid stands quite alone with the lowest ratios, whilst the alcohols, and especially methyl alcohol, have much higher ratios than the other compounds. Of the remaining substances the esters have higher values than the members of the first group.

There is little doubt that the low ratios for acetic acid are due to the existence of complex molecules in the saturated vapour at low temperatures. The densities of the saturated vapours of the alcohols, on the other hand, are normal at low temperatures, and complex molecules cannot, therefore, be present; but there is considerable evidence of their existence in the liquid state.

Ramsay and Shields (Phil. Trans. 184 A. p. 647; Trans. Chem. Soc. lxiii. p. 1089) have recently studied the surface-energy of a large number of compounds, and have described a method by which the molecular complexity of liquids may be ascertained. They show that most of the liquids investigated have the same molecular weight in the liquid as in the gaseous state, but that there is greater molecular complexity in the liquid state in the case of the fatty acids and alcohols. As regards homologous compounds—both acids and alcohols—the complexity diminishes with rise of molecular weight, and as regards individual compounds it diminishes with rise of temperature; in order of complexity methyl alcohol comes next to acetic acid.

These conclusions, based on totally different considerations, agree perfectly with those suggested in this paper and in the previous one on the same subject.

An explanation of the relatively high molecular volumes of the saturated vapour of the alcohols at low temperatures would be afforded by the assumption that complex molecules exist to some extent at the critical points, an assumption which is supported by the high critical densities.

It has been shown (Phil. Mag. Nov. 1890, p. 423) that if the generalizations of Van der Waals were strictly true, the following relation should be true for all substances :—

$$\frac{v}{v'}=\frac{V}{V'}=\frac{T}{T'}\times\frac{p'}{p},$$

where v and v' are the molecular volumes of saturated vapour, V and V′ those of liquid, and T and T′ the boiling-points on the absolute scale of temperature of any two substances at corresponding pressures p and p'. This relation should hold good at the critical point, or

$$\frac{PV}{T}=\text{constant},$$

where P, V, and T are the critical pressure, the critical molecular volume, and the absolute critical temperature of any substance. As the critical volumes are now known, it is possible to test this relation at the critical points of the various compounds, and the values of $\frac{PV}{T}$ are given in the table below.

Again, the ratios of the actual to the theoretical density (for a perfect gas) at the critical point should be the same for all substances if Van der Waals's generalizations were strictly true: these ratios are also given in the table under the heading $\frac{D}{D'}$.

Substance.	$\frac{PV}{T}$.	$\frac{D}{D'}$.
Fluorbenzene	16390	3·79
Chlorobenzene	16380	3·77
Bromobenzene	16270	3·80
Iodobenzene	16360	3·78
Benzene	16610	3·71
Carbon tetrachloride	16940	3·65
Stannic chloride	16630	3·76
Ether	16290	3·83
Methyl alcohol	13750	4·52
Ethyl alcohol	15470	4·02
Propyl alcohol	15460	4·02

Substance.	$\frac{PV}{T}$.	$\frac{D}{D'}$.
Acetic acid	12440	5·00.
Methyl formate	15840	3·93
Ethyl formate	15990	3·88
Methyl acetate	15750	3·94
Propyl formate	16070	3·87
Ethyl acetate	15730	3·95
Methyl propionate	15920	3·90
Propyl acetate	15760	3·94
Ethyl propionate	15830	3·92
Methyl butyrate	15900	3·90
Methyl isobutyrate	16070	3·87

Here, again, the grouping of the compounds is well marked: acetic acid has by far the lowest value of $\frac{PV}{T}$ (and the highest density ratio); the alcohols come next in order, the values for methyl alcohol standing about midway between those for the other alcohols and acetic acid; the esters agree well together, the values of $\frac{PV}{T}$ being in every case lower than for the members of the first group.

Discussion.

Prof. Ramsay was not sure that the existence of complexes would alter the molecular volume in the liquid state; for liquids seem very compact. Experiments on the surface-energy of liquids had proved that complex molecules do exist in the alcohols and acetic acid. Dr. Young's conclusion was, therefore, confirmed by experiments of an entirely different nature.

Prof. Herschel was gratified to see Van der Waals's theory so well borne out in liquids, and hoped to see it extended to solids. The recent researches of Prof. Roberts-Austen on alloys seemed to point in this direction.

Mr. Rogers said molecular complexes do exert an influence on the properties of substances, as had been shown by Prof. Thorpe's viscosity experiments. Van der Waals's generalizations should therefore be looked at from a chemical as well as from a physical point of view.

The President thought the numbers brought forward showed fair agreement, especially when it was remembered that Van der Waals took no account of the complex molecules. Contrary to Prof. Ramsay, he would rather expect aggregation to affect the molecular volumes in the liquid state, for only about one-fifth the space was supposed to be occupied by matter. On the other hand, the relatively small contraction of liquids on cooling did not support this view.

XXXI. *The Action of Electromagnetic Radiations on Films containing Metallic Powders. By* Professor G. M. MINCHIN, *M.A.*[*]

AT the last Edinburgh meeting of the British Association attention was directed by Dr. Dawson Turner to a discovery made by M. E. Branly, as to the effect produced on a glass tube filled with copper filings through the extremities of which tube are inserted two wires which dip into the filings (without, of course, touching each other inside the tube). The filings form, apparently, a continuous column of metal, the tube when held up to the light being opaque. Supporting the tube in a horizontal, or any other, position, and connecting its terminal wires with a galvanometer and a voltaic cell in a continuous circuit, the chances are that no current whatever will be indicated by the galvanometer. If such a current happens to exist, a very slight tap given to the tube or to its support will destroy the conductivity of the column of filings, and no current passes. Another such tap will cause the column to conduct, and so on.

At first sight it seems strange that such a metallic column should be a non-conductor; but, of course, when we see the result we can admit that the contact between every particle and its neighbours is of the microphonic kind, and thus that the whole column may be an extremely bad conductor.

The identity of the state of affairs in the tube with that in a microphone may be perceived by inserting a telephone in the circuit, and then we shall find that slight mechanical disturbances communicated to the column of filings will be accompanied by the rumbling noises which are heard in a telephonic circuit which contains a microphone.

---

[*] Read November 24, 1893.

Now the most striking peculiarity of such a column is that, while it is in the non-conducting state, if a spark is allowed to pass anywhere in its neighbourhood between the knobs of a Leyden jar or the poles of an induction-coil, the waves sent out from the spark at once render the column a conductor.

This experiment was reproduced at the meeting of the Physical Society on October 27 by Mr. Croft (*antea*, p. 413), and it at once struck me as bearing a strong resemblance to the effects produced by electromagnetic waves on photoelectric impulsion-cells. These cells I have already fully described (see Phil. Mag. March 1891) ; and in addition to what I showed with regard to them, I may here add that the effects of the sparks of an induction-coil on an impulsion-cell were observed when the cell (enclosed in a dark box) was over 140 feet distant from the coil and from everything connected with it. Moreover, it was found that, *in most cases*, the power of electromagnetic radiations to change the state of the cell from one of insensitiveness to one of sensitiveness to light depends on the length of the electromagnetic waves ; for, by adding capacity (by means of Leyden jars) to the sparking circuit, the waves ceased to affect the cell, and, on removing this capacity, their ability to effect the change was restored.

I have always held that this action of the cell is due to a rearrangement of the molecules on the sensitized surface of the plate in the cell by the electrical disturbances produced in the wires connected with its poles, these disturbances being due to the radiations emanating from the sparking circuit ; and I found that by completely enclosing the cell, *together with its terminal wires*, in a metal box, the ability of the radiations to act on the cell was destroyed.

After having seen the experiment shown by Mr. Croft, I filled several glass tubes with the filings of copper, tin, zinc, bismuth, antimony, &c., and found M. Branly's result in every case. It seems to me that if the filings are extremely fine (powders, in fact) the result is much more difficult to obtain. Powders used in this way are very great insulators, and they must be strongly pressed together by the corks which close the glass tubes in order to conduct; but then it becomes difficult to alter their conductivity by mechanical disturbances. On the other hand, a tube filled with very coarse iron filings was found to conduct under all circumstances; so that, apparently, the filings must be neither very fine nor very coarse.

As in the case of the impulsion-cells, the effect of the electromagnetic disturbances is cut off by completely enclosing the tubes and their terminal wires in a metal box. If the terminal wires are allowed to project outside the box—whether they are connected with the battery and galvanometer or not—the tubes are affected, which clearly shows that the result is due to the electrical disturbances set up in the leading wires of the tube.

In order to establish a closer connexion with the impulsion-cells, it seemed to me that films containing metallic particles almost in mathematical contact, but yet free to undergo very minute displacements of rotation, should replace the tubes of filings. Accordingly, I have formed such films by imbedding fine metallic powders in layers of gelatine and of collodion. The powders which I employ in these films are very much finer than the filings which I have used in the glass tubes.

The media, or bases, in which these powders are embedded are (so far as I have gone) gelatine and collodion.

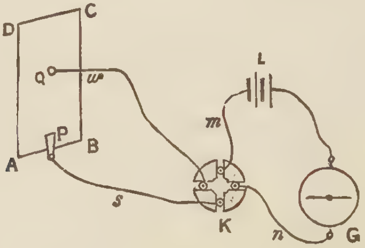
To form a gelatine film proceed thus:—Pour a very thin layer of dissolved gelatine over a glass or ebonite plate; let this layer get almost quite dry; if it gets quite dry, hold it over a beaker of boiling water until it absorbs a sufficient quantity of steam to render it very slightly plastic: take a test-tube with fine metallic powder immersed in alcohol; shake this up well, and rapidly pour some of it over the gelatine surface, securing, to all appearance, both continuity and uniformity of the metallic layer. The alcohol soon evaporates, leaving the metallic layer embedded in the gelatine, but not wholly sunk below it.

In the case of a collodion film, the metallic powder is shaken up in a test-tube and then rapidly poured over a glass plate. After some time the film dries and peels off the plate. The surface of the film which was in contact with the glass is the metallic surface sought ; the other surface of the film probably contains no metallic particles, and is simply a continuous surface of dry collodion which is an absolute non-conductor.

If the layer of gelatine above described is too thick, it will probably happen that its upper surface (that remote from the glass or ebonite plate) does not contain metallic particles, and is a non-conductor. In this case, if the film is peeled off the plate there remains on the plate a metallic surface which will answer our purpose.

Such a film may now replace the tube of filings in M. Branly's experiment.

Let ABCD represent the film, L a battery of one or two voltaic cells, G a galvanometer, and K a key which can be connected with the film by means of the wires s, w, the former clamped to the film at a point P, while the latter ends in a stout platinum piece the end of which is a round knob touching the film at a movable point Q.



Now it is found that when the contacts are all made no current passes, and we have to begin by making the platinum knob touch the film very close to P—perhaps $\frac{1}{2}$ millim. from P. Still no current passes, even when a spark of an induction-coil is allowed to pass a few feet from the film. But by touching either of the wires s, w with an electrified body (that which I commonly use is a common gas-lighting electrical machine), the resistance of the film and the contacts

P, Q is overcome, and a strong current is shown by the galvanometer. We can now gradually increase the distance between P and Q each time that Q is removed from P, the wire *s* or *w* being touched by the electrified body ; and by these successive steps the whole of the film is finally rendered conducting.

The special characteristic of such a film which I wish to point out is this : if, while the film is conducting, the current is stopped by breaking the contact at Q, the film will be found to be a non-conductor if the contact is almost instantly re-made at the same point Q ; while, if the contact is broken anywhere else in the circuit, as at K, and then re-made, the film will still be found to conduct.

Again, if when the film has been rendered a conductor, the circuit is broken at K, and left broken for (apparently) any length of time, and then re-made, the film will be found to be still a conductor.

The breaking of the contact at Q is instantly fatal to the conductivity of the film, if the film is not many days old ; but, after some days, it will be found that on breaking the contact at Q and then re-making it, the film remains a conductor, unless we delay the re-making of the contact for half a minute or so. Doubtless this is due to the fact that, as the film gets more hard and less plastic, the metallic particles find it more difficult to move, by rotation or otherwise, in their environment.

Finally, these films are very much less sensitive to mechanical disturbances given to their supports than the tubes of filings. They do not appear to be susceptible to the action of heat ; but in many instances their conductivity was destroyed by breathing upon them, or by allowing a stream of steam to strike their surfaces, the conductivity being always restored by the electromagnetic radiations. The more rapid vibrations of light have not produced any effect so far as my observations have gone.

The prime cause of the action is to be sought in the electrical surgings produced in the leads *s* and *w* ; but it is clear that the state of affairs at the places P, Q of contact of the film with the electrodes is a most important matter.

It seems clear, therefore, that the sensitized surface in an impulsion-cell has a close analogue in a slightly plastic film filled with almost mathematical completeness by fine metallic particles.

Discussion.

The President read a written communication from Prof. O. J. Lodge, in which the writer suggested that the phenomena of the films, and also of Lord Rayleigh's water-jet experiment (in which water drops are caused to coalesce by the presence of an electrified body), were due to the range of molecular attraction being increased by electric polarization.

Mr. Blakesley said he had tried Mr. Croft's experiments, and found that conductivity could be established in a tube of filings whilst the circuit was unclosed. Breaking the circuit of a transformer or electromagnet would produce conductivity ; hence he concluded that electric pressure from a cell was not essential. Another curious experiment was to put the discharging knobs of an electric machine on a photographic plate at a distance of a few inches. On turning the machine a small spark travels slowly along the plate from the negative to the positive knob. On reversing the polarity of the machine, the spark travels back along the same path, but if the polarity remains unchanged, a second spark usually travels along a different path.

Prof. C. V. Boys asked Prof. Minchin whether the films themselves, or the contacts between the electrode and film, are made conducting by the sparks.

Prof. S. P. Thompson wished to know if ordinary photographic dry plates would serve the purpose.

Mr. Evershed enquired whether the metal used as electrode made any difference.

Prof. Minchin, in his reply, maintained that the phenomena were due to electric impulses. He had not tried photographic plates, and had always used platinum for his electrodes.

XXXII. *On the Sudden Acquisition of Conducting-Power by a Series of Discrete Metallic Particles.* *By* Prof. OLIVER J. LODGE *.

THE recent experiments of Mr. Croft and Prof. Minchin remind me of an observation I frequently made when engaged with the syntonic arrangement of Leyden-jar circuits, or sympathetic electric resonance. I found, if the knobs of the receiver were very close together, a weak battery and bell being in circuit, that the occurrence of a scintilla at the receiver frequently caused the bell to ring for some time, and in general to show signs that the knobs were in a state of feebly adhesive contact. It was just as if their surface-layers or skins had been broken through, or opened out, in such a manner as to increase the molecular range of a few of the closest superficial molecules and thereby to cause cohesion to set in at a distance considerably greater than the ordinary distance.

A phenomenon which may be similarly caused is that discovered by Lord Rayleigh, with regard to the effect of electrified sealing-wax near a vertical water-jet: drops which would otherwise have rebounded being thereby caused to cohere. And, by the method of two impinging jets, the necessary difference of potential was shown to be only a volt or two, since a pair of impinging jets ceased to rebound if connected to the opposite terminals of a Grove cell.

Once more, the effect first observed by Mr. Guitard (1850), and rediscovered by myself and the late J. W. Clark, concerning the adhesion of dust-particles or mist-globules in electrified air, is but a more violent variety of the same sort of effect. And the action of electricity on a steam-jet discovered by Robert von Helmholtz and afterwards by Shelford Bidwell, and worked at by Richarz, Aitken, J. J. Thomson, and others, may not be very different.

Thinking of the Lord Rayleigh variety of experiment as in many respects the most definite, it is natural to explain it electrolytically as due to the polarization of the water-drops, or the formation in each drop of molecular chains each with a negatively charged oxygen atom at its termination on one hemisphere and a positively charged hydrogen atom at its opposite extremity.

---

* Read November 24, 1893, as a contribution to the discussion on Prof. Minchin's paper.

Such drops then colliding about the region of their poles would be attracted not only by their ordinary cohesive forces, but by electrical force also, and thus the effective molecular range would be increased and cohesion might set in over an unusual distance.

In some such way I have allowed myself to fancy that the adhesion of my knobs might be explained, and I suggest that the conductivity of a chain of metallic filings under an electric polarising influence may be due to something of the same cause.

That a tap should break the minute points of contact is likely enough, though I see no reason why another tap should restore communication,—if so it does.

At the last moment I write this and send it off in the hope that it may arrive in time to be read at the Meeting as a contribution to the discussion on these interesting experiments.

Nov. 23, 1893.

XXXIII. *On the Magnetic Shielding of Concentric Spherical Shells. By* A. W. Rücker, *F.R.S.*[*]

Special interest has of late attached to the problem of shielding magnetic forces by means of iron screens, and recently the Astronomer Royal has described the arrangement (designed by Messrs. Johnson and Phillips) by which he has protected the absolute magnetic instruments at Greenwich from the disturbances due to a dynamo.

In that case external space was protected against magnetic forces produced inside the enclosure.

The converse arrangement is sometimes adopted, and, as in Lord Kelvin's Marine Galvanometer, instruments are enclosed in iron and thus protected against external magnetic influence.

[*] Read November 24, 1893.

In the particular case when the iron shields are spherical shells, the method of attacking the mathematical problem involved is well known. Maxwell gives the working in the case of an internal space protected by a single shell (Ed. 3, vol. ii. p. 59), and also discusses the precisely analogous problem of the flow of currents interrupted by shells of different conductivity from the rest.

I believe, however, that a number of facts, which are of some practical importance, are not generally known, and I propose therefore to deal with them in the following paper, illustrating them by numerical examples.

In the earlier part of the paper the general theory is developed, chiefly for the sake of defining the notation used, though this course involves some recapitulation from Maxwell. The discussion of the best conditions for shielding, with which I mainly deal, is, I believe, new.

The principal point to which attention is directed is the calculation of the advantage gained by lamination, *i. e.* by using shields separated by air-gaps, like those employed by Mr. Christie, instead of a continuous mass of iron.

That lamination would be useful might be foreseen by the aid of the hydrodynamical analogue to the magnetic field.

If we suppose sources and sinks of equal power to be placed within an enclosure, we may weaken the flow outside, either by making the enclosure or part of it very permeable, or by making the bounding surface very impermeable.

In like manner in the magnetic problem there are two ways of reducing the internal field, viz. by enclosing the central region by shells of greater, or by shells of less, permeability than the surrounding space. Although we cannot in practice use any material of less permeability than the air, we can introduce into an iron shell layers of material less permeable than itself. These may tend to reduce the external field by preventing as it were the force from penetrating to the external shell.

This analogy has long suggested that laminated shells, in which both the permeability of the iron and the relative impermeability of other materials are utilized, may be the most effective shields; and we are thus led to enquire which is the best arrangement that can be made under specified conditions as to the magnitude of the space at our disposal or as to the weight of the shielding material used. The answer to this question is supplied by the following discussion.

As the formulæ are somewhat heavy, I shall for the most part confine myself to shells formed of the same material and separated by air-gaps.

The second problem, to the result of which I think it is desirable to draw attention, is that which defines the relations between the shielding exerted by a number of concentric shells, (1) on external space when the magnetic forces are produced within the shells; and (2) on the enclosed space when the magnetic forces are produced outside. A full discussion of this is given below.

The whole of the investigation is subject to three limitations. I have confined myself:—

(1) to concentric shells;

(2) to cases in which the equipotential surfaces are surfaces of revolution about a line through the common centre of the shells;

(3) to the case in which the permeability of each shell is constant.

As regards the first two limitations, I think it will be seen that the conclusions arrived at are capable of generalization in such a way as to enable a better approach to be made to good shielding arrangements than would be the case if the results of the discussion in the simple case of spherical shells were unknown. The third limitation no doubt affects the applicability of the formulæ to practice. In spite of this, however, I venture to think that they afford some useful guidance, and that at all events they help to put the practical problem in definite terms.

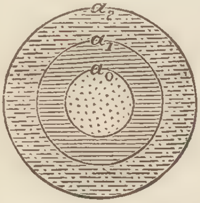
The Relation between the Shielded and Unshielded Fields when the Shielded Space is (1) *within,* (2) *without the Shielding Shells.*

Take the common centre of the shells as origin. Let the potential within any shell be expanded in terms of zonal spherical harmonics, then the terms corresponding to P_n will be of the form

$$\left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n.$$

Each such term can be discussed independently, and it is more convenient to make the subscript numbers refer to the shell than to the order of the harmonic.

If, therefore, there are m shells, let the radii of the surfaces of separation be a_0, a_1, a_2, &c. a_m. Then a_0 is the radius of the central space, a_1, a_2, &c. are the external radii of consecutive shells (see figure).



Let $\mu_0, \mu_1, \dots \mu_m$ be the permeabilities of the enclosed space and of the shells, and for generality let M be the permeability of external space.

Let the coefficient of P_n in the expansion of the potential in the pth shell be

$$\phi_p r^n + \psi_p / r^{n+1}.$$

Then at the boundary between the pth and the $(p+1)$th shells (of which the radius is a_p) the following conditions must be fulfilled:—

$$\phi_p a_p^n + \psi_p / a_p^{n+1} = \phi_{p+1} a_p^n + \psi_{p+1} / a_p^{n+1},$$

and

$$\mu_p \{ n\phi_p a_p^{n-1} - (n+1)\psi_p / a_p^{n+2} \}$$
$$= \mu_{p+1} \{ n\phi_{p+1} a_p^{n-1} - (n+1)\psi_{p+1} / a_p^{n+2} \};$$

or if we write

$$\mathrm{N} = (n+1)/n, \qquad \alpha_p = a_p^{-(2n+1)},$$

we get $$\phi_p + \psi_p \alpha_p - \phi_{p+1} - \psi_{p+1} \alpha_p = 0,$$

and $$\mu_p \phi_p - \mathrm{N}\mu_p \alpha_p \psi_p - \mu_{p+1} \phi_{p+1} + \mathrm{N}\mu_{p+1} \alpha_p \psi_{p+1} = 0.$$

If we suppose, for the sake of symmetry, and without reference for the moment to the expression of physical facts, that the coefficients of P_n in the potentials in the central enclosed space and in external space are

$$\phi_0 r^n + \psi_0 / r^{n+1} \quad \text{and} \quad \Phi r^n + \Psi / r^{n+1}$$

respectively, we have in all $2m+4$ quantities to deal with, and since the m shells have $m+1$ bounding surfaces, there are $2m+2$ equations between them, all of which are linear and of the types given above.

We thus get a series of equations as follows :—

$$\left.\begin{aligned}
&\Phi+\Psi\alpha_m-\phi_m-\psi_m\alpha_m=0.\\
&\mathrm{M}\Phi-\mathrm{MN}\Psi\alpha_m-\mu_m\phi_m+\mu_m\mathrm{N}\psi_m\alpha_m=0.\\
&\phi_m+\psi_m\alpha_{m-1}-\phi_{m-1}-\psi_{m-1}\alpha_{m-1}=0.\\
&\mu_m\phi_m-\mu_m\mathrm{N}\psi_m\alpha_{m-1}-\mu_{m-1}\phi_{m-1}+\mu_{m-1}\mathrm{N}\psi_{m-1}\alpha_{m-1}=0.\\
&\qquad\text{\&c.,}\qquad\qquad\qquad\qquad\text{\&c.}\\
&\phi_2+\psi_2\alpha_1-\phi_1-\psi_1\alpha_1=0.\\
&\mu_2\phi_2-\mu_2\mathrm{N}\psi_2\alpha_1-\mu_1\phi_1+\mu_1\mathrm{N}\psi_1\alpha_1=0.\\
&\phi_1+\psi_1\alpha_0-\phi_0-\psi_0\alpha_0=0.\\
&\mu_1\phi_1-\mu_1\mathrm{N}\psi_1\alpha_0-\mu_0\phi_0+\mu_0\mathrm{N}\psi_0\alpha_0=0.
\end{aligned}\right\}(1)$$

In the case of shielding an enclosed space the potential of the inducing forces can be expanded within and in the neighbourhood of the shells in terms of positive powers of r, and since the potential of the induced magnetization must be finite when $r=0$, this also is included in ϕ_0. Hence $\psi_0=0$. In external space the expansion of the potential contains both positive and negative terms, the former due to the inducing, and the latter to the induced magnetization.

As the inducing forces are given Φ is known.

As $\psi_0=0$ and Φ is given, the number of unknowns is reduced by 2 and is equal to the number of the equations.

If, on the other hand, the inducing forces are produced in the central space within the shell, their potential must be expanded in inverse powers of r if the series is to be convergent at the boundary of the enclosure and beyond it. Hence ψ_0 is known. In external space the terms due both to the induced and inducing magnetizations are expressed in negative powers, and are included in Ψ. Thus $\Phi=0$.

Now ϕ_0/Φ is the ratio of the shielded to the unshielded field in the first case, in so far as it depends on the potential term under consideration, and Ψ/ψ_0 is the corresponding quantity when the forces are produced within the shell.

The relation between these may now be found.

Putting $\psi_0=0$ and writing Δ for the determinant formed by the coefficients of the ϕ's and ψ's in the above equations, except those which occur in the first and last columns, we have :—

$$\Phi\begin{vmatrix} 1, & \alpha_m, & -1, & -\alpha_m, \ldots\ldots\ldots & 0 \\ M, & -MN\alpha_m, & -\mu_m, & \mu_m N\alpha_m, \ldots\ldots\ldots & 0 \\ 0, & 0, & 1, & \alpha_{m-1}, \ldots\ldots\ldots & 0 \\ 0, & 0, & \mu_m, & -\mu_m N\alpha_{m-1}, \ldots\ldots & 0 \\ & & \ldots\ldots\ldots\ldots & & \\ 0, & 0, & 0, & 0, \ldots\ldots\ldots & -\alpha_1 \\ 0, & 0, & 0, & 0, \ldots\ldots & \mu_1 N\alpha_1 \\ 0, & 0, & 0, & 0, \ldots\ldots\ldots & \alpha_0 \\ 0, & 0, & 0, & 0, \ldots\ldots & -\mu_1 N\alpha_0 \end{vmatrix} = \phi_0\Delta.$$

This may be written

$$\frac{\Phi\alpha_m}{M}\begin{vmatrix} M, & M, & -M, \ldots\ldots\ldots & 0 \\ M, & -MN, & -\mu_m, \ldots\ldots\ldots & 0 \\ 0, & 0, & 1, \ldots\ldots\ldots & 0 \\ & \ldots\ldots\ldots\ldots & & \\ 0, & 0, & 0, \ldots & -\mu_1 N\alpha_0 \end{vmatrix} = \phi_0\Delta\ ;$$

whence, by subtracting the first from the second row,

$$\frac{\Phi\alpha_m}{M}\begin{vmatrix} M, & M, & -M \ldots\ldots\ldots & 0 \\ 0, & -M(N+1), & M-\mu_m, \ldots\ldots & 0 \\ 0, & 0, & 1, \ldots\ldots\ldots & 0 \\ & \ldots\ldots\ldots\ldots & & \\ 0, & 0, & 0, \ldots & -\mu_1 N\alpha_0 \end{vmatrix} = \phi_0\Delta.$$

$$\therefore\ -\Phi\alpha_m(N+1)M\begin{vmatrix} 1, & \alpha_{m-1}, \ldots\ldots\ldots & 0 \\ \mu_m, & -\mu_m N\alpha_{m-1}, \ldots\ldots & 0 \\ & \ldots\ldots\ldots & \\ 0, & 0, \quad 0, \ldots & -\mu_1 N\alpha_0 \end{vmatrix} = \phi_0\Delta.$$

Proceeding in the same way with the new determinant, we finally get :—

$$\{-(N+1)^{m+1}\Phi\alpha_m\alpha_{m-1}\ldots\alpha_0 M\mu_m\mu_{m-1}\ldots\mu_1=\phi_0\Delta.$$

Next, putting $\Phi=0$, we get in like manner

$$\Psi\Delta = \{-(\mathrm{N}+1)\}^{m+1}\psi_0\alpha_m\alpha_{m-1}\ldots\alpha_0\mu_m\mu_{m-1}\ldots\mu_1\mu_0\psi_0;$$

$$\therefore\quad \frac{\phi_0}{\Phi\mathrm{M}} = \frac{\Psi}{\mu_0\psi_0}.$$

Hence if $\mathrm{M}=\mu_0$, *i.e.* if the permeabilities of the enclosed and the external space are the same, the ratios of the shielded to the unshielded fields are the same *for each harmonic term* whether the shielded field be external or internal.

The most interesting application of this proposition is to the first harmonic term.

In this case the ϕ_0 and Φ are coefficients of terms of the form $r\mathrm{P}_1$, that is the field is uniform; and ψ_0 and Ψ are coefficients of terms of the form P_1/r^2, that is, the field is produced by a small magnet placed at the centre of the shells.

Hence the shielding effect on external space when a small magnet is placed at the centre of the shells is the same as the shielding effect on the enclosed space when the shells are placed in a uniform field.

In what follows I shall for the most part suppose that the magnetic forces are produced within the shells; but the above result enables the conclusions arrived at to be applied to the case of shielding against external forces.

(1) *Case of a Single Shell when the Permeabilities of the Internal and External Space are Unity.*

This case is well known, but may be included for the sake of completeness.

Since $\mu_0=\mathrm{M}=1$, the equations (1) reduce to

$$\phi_1+\psi_1\alpha_1-\Psi\alpha_1 = 0,$$

$$\mu_1\phi_1-\mathrm{N}\mu_1\alpha_1\psi_1+\mathrm{N}\alpha_1\Psi = 0,$$

$$\phi_0+\psi_0\alpha_0-\phi_1-\psi_1\alpha_0 = 0,$$

$$\phi_0-\mathrm{N}\alpha_0\psi_0-\mu_1\phi_1+\mathrm{N}\mu_1\alpha_0\psi_1 = 0.$$

Hence Ψ/ψ_0 is the ratio of the shielded to the unshielded potential or, since this ratio is everywhere constant, that of the shielded to the unshielded field, in so far as it depends on the term under consideration, is

$$\frac{\alpha_0\mu_1(\mathrm{N}+1)^2}{(\mathrm{N}\mu_1+1)(\mathrm{N}+\mu_1)\alpha_0-\mathrm{N}\alpha_1(\mu_1-1)^2}. \quad . \quad . \quad . \quad (2)$$

If the unshielded field is due to a small magnet placed at the centre of the sphere, the only term in the expression for the potential is

$$\mathrm{A}_1\mathrm{P}_1/r^2.$$

Hence

$$n=1, \quad \mathrm{N}=(n+1)/n=2, \quad \alpha_0=a_0^{-(2n+1)}=a_0^{-3}, \quad \&c.$$

Then the above ratio becomes

$$\frac{9a_1{}^3\mu_1}{(2\mu_1+1)(2+\mu_1)a_1{}^3-2(\mu_1-1)^2a_0{}^3}=\frac{9a_1{}^3\mu_1}{9a_1{}^3\mu_1+2(\mu_1-1)^2(a_1{}^3-a_0{}^3)}.$$

If μ_1 is so great that a small integer may be neglected with regard to it, this reduces to

$$\frac{9}{2\mu_1}\frac{a_1{}^3}{a_1{}^3-a_0{}^3};$$

but this approximate form is only valid if a_1 and a_0 are not nearly equal.

It will be observed that the shielding depends only on the ratio of $a_1 : a_0$, and not on the absolute dimensions.

If the shell is thin, let $a_0=a_1-t$. Then, if squares and higher powers of t are neglected,

$$\frac{\Psi}{\psi_0}=\frac{3\mu_1}{3\mu_1+2(\mu_1-1)^2\dfrac{t}{a_1}}.$$

If μ_1 is not large this may be written

$$\frac{\Psi}{\psi_0}=1-\frac{2(\mu_1-1)^2}{3\mu_1}\cdot\frac{t}{a_1}.$$

If μ_1 is large it becomes

$$\frac{\Psi}{\psi_0}=\frac{1}{1+\frac{2}{3}\mu_1\dfrac{t}{a_1}}.$$

In this approximation we have neglected terms of the order $\mu_1 t^2/a_1^2$. The expression

$$\frac{\Psi}{\psi_0} = 1 - \frac{2}{3}\mu_1 \frac{t}{a_1}$$

is only valid if terms of the order $\mu_1^2 t^2/a_1^2$ are negligible ; and as μ_1 is by supposition large, they may be very much greater than those which have already been rejected.

As an example, if $t/a_1 = 0{\cdot}01$, the ratio of the shielded to the unshielded field is 3/13 if $\mu_1 = 500$, and 3/23 if $\mu_1 = 1000$.

The following Table of Values is calculated for cases in which $a_1 - a_0$ is not small, on the assumption that $\mu_1 = 1000$. The corresponding numbers can be obtained in any other case when μ_1 is large by remembering that the ratio under investigation varies inversely as μ_1.

a_0/a_1.	Ψ/ψ_0.
0·9	1/60
0·8	1/108
0·7	1/146
0·6	1/174
0·5	1/194

This table illustrates the difficulty of improving the shielding by increasing the thickness of a shield already moderately thick. When the latter is increased in the ratio 5 : 1, and the internal empty space is reduced sixfold, the shielded field is only reduced in the ratio of about 1 : 3.

In general, if the volume of the material employed in a shell is seven times that of the internal spherical space the external field will be reduced to 1/100, 1/200, or 1/300 of its unshielded value, according as the permeability of the material is 500, 1000, or 1500.

Next consider the case of a small magnet placed at a distance b from the centre of the shell.

If the axis of the magnet lies on a diameter, and if the moment is ψ_0, the unshielded potential is

$$\psi_0 \left\{ \frac{P_1}{r^2} + \frac{2P_2 b}{r^3} + \frac{3P_3 b^2}{r^4} + \&c. + \frac{nP_n b^{n-1}}{r^{n+1}} + \&c. \right\},$$

where r is measured from the centre of the sphere.

To find the potential in the shielded field, we must multiply each term in the expression by the corresponding value of

$$\frac{\alpha_0\mu_1(\mathrm{N}+1)^2}{(\mathrm{N}\mu_1+1)(\mathrm{N}+\mu_1)\alpha_0-\mathrm{N}\alpha_1(\mu_1-1)^2},$$

which may be called the *shielding factor.*

If the shell is thin, so that $a_0=a_1-t$, where t is small, this factor (remembering that $\alpha_0=a_0^{-(2n+1)}$) becomes

$$\frac{a_1^{2n+1}\mu_1(\mathrm{N}+1)^2}{a_1^{2n+1}\mu_1(\mathrm{N}+1)^2+(2n+1)\mathrm{N}(\mu_1-1)^2a_1^{2n}t}$$

$$=\frac{(2n+1)\mu_1}{(2n+1)\mu_1+n(n+1)(\mu_1-1)^2t/a_1},$$

which, if μ_1 is large, simplifies to

$$\frac{2n+1}{2n+1+n(n+1)\mu_1t/a_1}.$$

Calling these shielding factors S_1, S_2, &c., the unshielded and shielded forces in the direction of the radius are given by

$$-\psi_0\left\{\frac{2\mathrm{P}_1}{r^3}+\frac{2\,.\,3\mathrm{P}_2b}{r^4}+\frac{3\,.\,4\,.\,\mathrm{P}_3b^2}{r_5}+\&\mathrm{c.}\right\}$$

and by

$$-\psi_0\left\{\frac{2\mathrm{P}_1\mathrm{S}_1}{r^3}+\frac{2\,.\,3\,.\,\mathrm{P}_2b\mathrm{S}_2}{r^4}+\frac{3\,.\,4\,.\,\mathrm{P}_3b^2\mathrm{S}_3}{r^5}+\&\mathrm{c.}\right\}$$

respectively.

If we consider any point on the diameter on which the axis of the magnet lies and on the side towards which it is moved, all the P's are equal to unity.

The larger b becomes, the more important are the terms in which the shielding factors S_2, S_3 are small, and the more efficient is the shielding. The total force is, however, increased, as all these terms are added to the first. In other words, the force is increased owing to the approach of the magnet to the point considered, and it is only the addition to its original value which is better shielded. The average shielding is improved, but the force is increased.

The converse statement holds good for the side from which the magnet is moved. In that case the P's are alternately

-1 and $+1$, the shielding is less efficient, but the force is diminished.

It may be worth while to illustrate this by a numerical example. Let the magnet be placed on a diameter at a distance from the centre = half the external radius; and let the forces be calculated at points on the same diameter, the distances of which from the exterior of the shell are also half the external radius.

Then

$$b = a_1/2, \quad r = 3a_1/2.$$

Let the thickness of the shield be 0·01 of the radius, so that $t/a_1 = 0{\cdot}01$, and let the permeability be 500.

The unshielded force is :—

$$-\frac{\psi_0}{a_1^3}\{0{\cdot}593\,P_1 + 0{\cdot}593\,P_2 + 0{\cdot}395\,P_3 + 0{\cdot}219\,P_4 + 0{\cdot}100\,P_5 + 0{\cdot}051\,P_6 + 0{\cdot}023\,P_7 + 0{\cdot}010\,P_8 + \&c.\}.$$

Multiplying each term by the corresponding shielding factor, the shielded force is :—

$$-\frac{\psi_0}{a_1^3}\{0{\cdot}137\,P_1 + 0{\cdot}085\,P_2 + 0{\cdot}041\,P_3 + 0{\cdot}018\,P_4 + 0{\cdot}008\,P_5 + 0{\cdot}003\,P_6 + 0{\cdot}0011\,P_7 + 0{\cdot}0004\,P_8\}.$$

The true values of the unshielded forces are

$$-2\psi_0/a_1^3 \quad \text{and} \quad 2\psi_0/(2a_1)^3 = 0{\cdot}25\psi_0/a_1^3.$$

The above series gives

$$-1{\cdot}994\psi_0/a_1^3 \quad \text{and} \quad 0{\cdot}248\psi_0/a_1^3,$$

which proves that the number of terms taken gives results sufficiently approximate for the purpose of illustration.

The corresponding values of the shielded forces are

$$-0{\cdot}294\psi_0/a_1^3 \quad \text{and} \quad 0{\cdot}081\psi_0/a_1^3.$$

These conclusions may be summed up as follows :—

Let A and B be the two external points, and let A be that towards which the magnet is moved. Then we get the results given in the following table :—

Position of Magnet.	Numbers proportional to force when		Ratio.
	Unshielded.	Shielded.	
Centre	0·593 at A & B	0·137	0·23
Halfway between centre and outer boundary of shell.	2 at A	0·294	0·15
	0·25 at B	0·081	0·33

Case of Two or Three Concentric Shells of the same Permeability separated by Air-Gaps.

In the case of three shells there are twelve simultaneous equations, and we finally get :—

$$\left.\begin{aligned}\Psi\,[\xi\eta(\eta\alpha_1-\xi\alpha_0)\alpha_1\alpha_4(\alpha_4-\alpha_5)(\alpha_2-\alpha_3)\\ +\xi\eta(\eta\alpha_2-\xi\alpha_3)\alpha_2\alpha_4(\alpha_0-\alpha_1)(\alpha_4-\alpha_5)\\ +\xi\eta(\eta\alpha_5-\xi\alpha_4)\alpha_2\alpha_3(\alpha_0-\alpha_1)(\alpha_2-\alpha_3)\\ +\alpha_1\alpha_3(\xi\alpha_3-\eta\alpha_3)(\xi\alpha_4-\eta\alpha_5)(\xi\alpha_0-\eta\alpha_1)]\\ =\psi_0\alpha_0\alpha_1\alpha_2\alpha_3\alpha_4(\mathrm{N}+1)^6\mu^3,\end{aligned}\right\}\quad . \quad . \quad . \quad (3)$$

where $\xi=(\mathrm{N}\mu+1)(\mathrm{N}+\mu)$ and $\eta=\mathrm{N}(\mu-1)^2$.

This reduces to the case of two shells if we put any two consecutive α's equal to each other ; and it will be found that in all these cases the expression assumes the same form.

If $\alpha_4=\alpha_5$, remembering that $\xi-\eta=\mu(\mathrm{N}+1)^2$,

$$\left.\begin{aligned}\Psi\{-\xi\eta\alpha_2(\alpha_0-\alpha_1)(\alpha_2-\alpha_3)+\alpha_1(\xi\alpha_2-\eta\alpha_3)(\xi\alpha_0-\eta\alpha_1)\}\\ =\psi_0\alpha_0\alpha_1\alpha_2\mu^2(\mathrm{N}+1)^4\\ =\Psi\mathrm{D}\text{ say.}\end{aligned}\right\}\quad . \quad (4)$$

Then

$$\left.\begin{aligned}\frac{\mathrm{D}^2d\Psi}{\psi_0\mu^2(\mathrm{N}+1)^4}=&\{\xi\eta\alpha_2(\alpha_2-\alpha_3)-\eta\alpha_1(\xi\alpha_2-\eta\alpha_3)\}\alpha_1{}^2\alpha_2d\alpha_0\\ &+\{-\xi\eta\alpha_2\alpha_0(\alpha_2-\alpha_3)+\eta\alpha_1{}^2(\xi\alpha_2-\eta\alpha_3)\alpha_0\alpha_2d\alpha_1\\ &+\{\xi\eta\alpha_2{}^2(\alpha_0-\alpha_1)-\eta\alpha_1\alpha_3(\xi\alpha_0-\eta\alpha_1)\}\alpha_0\alpha_1d\alpha_2\\ &+\{-\xi\eta\alpha_2(\alpha_0-\alpha_1)+\eta\alpha_1(\xi\alpha_0-\eta\alpha_1)\}\alpha_0\alpha_1\alpha_2d\alpha_3.\end{aligned}\right\}\ (5)$$

As α_0 and α_3 do not occur in the coefficients of $d\alpha_0$ and $d\alpha_3$ respectively, it is evident that no particular values of either these quantities give a maximum or minimum value of Ψ when the other three α's are arbitrarily selected.

Again, the coefficient of $d\alpha_0$ may be written

$$\eta\alpha_1^2\alpha_2\{\alpha_2(\xi\alpha_2-\xi\alpha_3)-\eta\alpha_1(\xi\alpha_2-\eta\alpha_3)\}.$$

Now $\alpha_2<\alpha_1$, and since ξ is $>\eta$,

$$\xi\alpha_2-\xi\alpha_3 \text{ is } <\xi\alpha_2-\eta\alpha_3.$$

Hence this expression is always negative, *i. e.* Ψ diminishes as α_0 increases, that is as the radius a_0 diminishes.

Thus adding the permeable material within the shell so as to reduce the internal cavity always improves the shielding.

In like manner, by discussing the coefficient of $d\alpha_3$, it may be shown that an improvement is always effected by increasing the external diameter.

If, however, the smallest and largest radii are given, Ψ may be a maximum or minimum if the coefficients of $d\alpha_1$ and $d\alpha_2$ vanish; *i. e.* if

$$\alpha_1^2=\frac{\xi\alpha_2\alpha_0(\alpha_2-\alpha_3)}{\xi\alpha_2-\eta\alpha_3},$$

and

$$\alpha_2^2=\frac{\alpha_1\alpha_3(\xi\alpha_0-\eta\alpha_1)}{\xi(\alpha_0-\alpha_1)}.$$

Let us first suppose that α_1 is also given, *i. e.* that the dimensions of the inner shell are fixed, then the equations enable us to determine the effect of adding another external shell of which the outer radius is also given.

We may first suppose the external shell to be of vanishing thickness, in which case $\alpha_2=\alpha_3$, and then allow α_2 to increase, *i. e.* the inner radius of the outer shell (a_2) to diminish.

If $\alpha_2=\alpha_3$ the coefficient of $d\alpha_2$ is negative, for $\alpha_3<\alpha_1$ and $\xi(\alpha_0-\alpha_1)<\xi\alpha_0-\eta\alpha_1$.

Hence the addition of a thin external shell reduces Ψ, *i. e.* improves the shielding.

This improvement will reach a maximum when

$$\alpha_2^2=\frac{\alpha_1\alpha_3(\xi\alpha_0-\eta\alpha_1)}{\xi(\alpha_0-\alpha_1)};$$

but in order that this may correspond to a physical reality we must have

$$\alpha_1 > \alpha_2 > \alpha_3.$$

The first of these inequalities when combined with the above equation is equivalent to

$$\alpha_1 \xi(\alpha_0 - \alpha_1) > \alpha_3(\xi\alpha_0 - \eta\alpha_1).$$

As α_1 is $>\alpha_3$ and $\xi(\alpha_0 - \alpha_1) < \xi\alpha_0 - \eta\alpha_1$, this condition may be, but is not necessarily, fulfilled.

The second is equivalent to

$$\alpha_1(\xi\alpha_0 - \eta\alpha_1) > \alpha_3 \xi(\alpha_0 - \alpha_1),$$

which is always true.

Hence we conclude that if the external radius of the outer shell is so chosen that

$$\alpha_3 < \alpha_1 \frac{\xi(\alpha_0 - \alpha_1)}{\xi\alpha_0 - \eta\alpha_1},$$

there will be a maximum value of the shielding, while an air-space intervenes between the two shields. There will not be a maximum if α_3 is greater than this limit. In the case of the first harmonic the condition for a maximum becomes

$$a_3 > a_1 \left\{ \frac{\xi a_1^3 - \eta a_0^3}{\xi(a_1^3 - a_0^3)} \right\}^{\frac{1}{3}};$$

where a_0, a_1, and a_3 are the radii of the corresponding surfaces. If there is such a maximum it is evident that for some thickness of the external shell less than that which gives the best result the shielding must be the same as if the whole of the air-gap were filled.

To find this value of α_2 we have to put $\alpha_2 = \alpha_1$ in the expression for Ψ, and equate the result to the value of Ψ when α_2 is not $= \alpha_1$.

Remembering that $\mu(\mathrm{N}+1)^2 = \xi - \eta$ we get

$$\frac{\alpha_0}{(\xi\alpha_0 - \eta\alpha_3)(\xi - \eta)} = \frac{\alpha_0\alpha_1\alpha_2}{-\xi\eta\alpha_2(\alpha_0 - \alpha_1)(\alpha_1 - \alpha_3) + \alpha_1(\xi\alpha_2 - \eta\alpha_3)(\xi\alpha_0 - \eta\alpha_1)}.$$

This leads to a quadratic in α_2, of which one root must evidently be $\alpha_2 = \alpha_1$.

The other root is

$$\alpha_2=\alpha_3\frac{\xi\alpha_0-\eta\alpha_1}{\xi(\alpha_0-\alpha_1)}.$$

If then we write $I=(\xi\alpha_0-\eta\alpha_1)/\xi(\alpha_0-\alpha_1)$, where I is a function only of the dimensions of the inner shell, of the permeability of the material and of the order of the harmonic term considered, we have reached the following conclusions.

If $\alpha_3 I>\alpha_1$, the shielding improves continuously as the thickness of the external shell increases from without inwards.

If $\alpha_3 I<\alpha_1$, the shielding at first improves and then deteriorates as the thickness increases.

When $\alpha_2=\alpha_3 I$, the shielding is the same as when the whole of the space between the shells is filled, *i. e.* as when $\alpha_2=\alpha_1$.

When $\alpha_2{}^2=\alpha_1\alpha_3 I$, the shielding is a maximum, after which it diminishes until $\alpha_2=\alpha_1$.

In the particular case when the maximum shielding is equal to that when the hollow space is just filled, the conditions $\alpha_2=\alpha_3 I$ and $\alpha_2{}^2=\alpha_1\alpha_3 I$ must be simultaneously fulfilled. Hence $\alpha_2=\alpha_1$, and there is only one maximum, which occurs when the hollow space is just filled.

This can only be the case when

$$\alpha_2{}^2=\alpha_1{}^2=\frac{\alpha_1\alpha_3(\xi\alpha_0-\eta\alpha_1)}{\xi(\alpha_0-\alpha_1)},$$

i. e., when

$$\xi\alpha_1{}^2-\alpha_1(\xi\alpha_0+\eta\alpha_3)+\xi\alpha_0\alpha_3=0.$$

It may be worth while to give a numerical example to illustrate these results.

Let the small magnet be placed in the centre. Then the only harmonic term is that for which $n=1$ and $N=2$.

Let $\alpha_0=8\alpha_3=2\alpha,$

i. e. $a_3=2a_0$ and $a_1=1{\cdot}260a_0.$

Let $\mu=501.$

Then

$$\xi=(N\mu+1)(N+\mu)=1003\times 503,$$

$$\eta=N(\mu-1)^2=2\times 500^2=\xi \text{ nearly};$$

or to a closer approximation,

$$\eta=\xi(1-0{\cdot}009).$$

Hence

$$\mathrm{I}=\frac{\xi\alpha_0-\eta\alpha_1}{\xi(\alpha_0-\alpha_1)}=1+0{\cdot}009.$$

By substituting in (4) we get the values of Ψ/ψ_0 given in the following table for different values of α_2. The volumes are expressed in terms of the space enclosed by the innermost surface.

$a_2=$	$a_2=$	Volume of outer shell.	Total Volume.	Ψ/ψ_0.
$a_3=a_0/8.$	$a_3=2a_0.$	0	1·00	0·018
$1{\cdot}009a_3=0{\cdot}1261a_0.$	$0{\cdot}997a_3=1{\cdot}994a_0.$	0·07	1·07	0·0102
$3a_3/2=3a_0/16.$	$0{\cdot}873a_3=1{\cdot}746a_0.$	2·66	3·66	0·0007
$2a_3\doteq a_0/4.$	$0{\cdot}794a_3=1{\cdot}588a_0.$	4·00	5·00	0·0006
$3a_3=3a_0/8.$	$0{\cdot}693a_3=1{\cdot}386a_0.$	5·33	6·33	0·0009
$a_1=4a_3=a_0/2.$	$0{\cdot}630a_3=1{\cdot}260a_0.$	6·00	7·00	0·0102

Turning next to the case when α_1 and α_2 both vary, we may determine the maximum shielding for given external and internal radii.

The conditions to be fulfilled are:

$$\left.\begin{aligned}-\xi\alpha_2\alpha_0(\alpha_2-\alpha_3)+\alpha_1^2(\xi\alpha_2-\eta\alpha_3)=0\\ \xi\alpha_2^2(\alpha_0-\alpha_1)-\alpha_1\alpha_3(\xi\alpha_0-\eta\alpha_1)=0\end{aligned}\right\}\quad . \quad . \quad (6)$$

Adding these we get either

$$\alpha_2=\alpha_1 \quad \text{or} \quad \alpha_2\alpha_1=\alpha_0\alpha_3.$$

The first has reference only to the point at which the maximum value is equal to that produced when all the space between the shells is filled.

The latter leads to the equation

$$\xi\alpha_2^3(\alpha_2-\alpha_3)-\alpha_3^2\alpha_0(\xi\alpha_2-\eta\alpha_3)=0.$$

This equation is a biquadratic in α_2.

Since the expression is positive when $\alpha_2=-\infty$ and negative when $\alpha_2=\alpha_3$, there is at least one real root $<\alpha_3$, with which, since $\alpha_2>\alpha_3$, we are not concerned.

As regards roots $>\alpha_3$, it is evident, since the whole expression is positive when $\alpha_2=\infty$, that there is at least one real positive root $>\alpha_3$. Also, as $\xi\alpha_2-\eta\alpha_3$ is positive, that there can only be one positive root $>\alpha_3$.

This root must be $<\alpha_1$, *i. e.* since $\alpha_2\alpha_1=\alpha_0\alpha_3$, $\alpha_2^2<\alpha_0\alpha_3$; *i. e.* the expression must be positive when $\alpha_2^2=\alpha_0\alpha_3$.

This is equivalent to the condition

$$\xi\sqrt{\alpha_0}(\sqrt{\alpha_0}-\sqrt{\alpha_3})>\sqrt{\alpha_3}(\xi\sqrt{\alpha_0}-\eta\sqrt{\alpha_3}).$$

As $\qquad \alpha_0>\alpha_3$ and $\xi\sqrt{\alpha_0}-\xi\sqrt{\alpha_3}<\xi\sqrt{\alpha_0}-\eta\sqrt{\alpha_3}$,

this equality may be, but is not necessarily, satisfied.

Before discussing these equations further, it is convenient to modify their form.

Let

$$\frac{\alpha_0}{\alpha_1}=\frac{\alpha_2}{\alpha_3}=\lambda\,; \qquad \eta=\xi(1-\epsilon)\,; \qquad \frac{\alpha_0}{\alpha_3}=\mathrm{L}.$$

Then from (6),

$$\lambda^3(\lambda-1)-\mathrm{L}(\lambda-1+\epsilon)=0, \quad . \quad . \quad . \quad . \quad (7)$$

and from (4),

$$\Psi/\psi_0=\frac{\mu^2(\mathrm{N}+1)^4\lambda^2\mathrm{L}}{\xi^2\{(\lambda-1+\epsilon)^2\mathrm{L}-(1-\epsilon)\lambda^2(\lambda-1)^2\}}\,; \qquad (7a)$$

or from (7),

$$\Psi/\psi_0=\frac{\mu^2(\mathrm{N}+1)^4\mathrm{L}^2}{\xi^2(\lambda-1)^2\{\lambda^4-\mathrm{L}(1-\epsilon)\}}=\frac{\mathrm{L}^2\epsilon^2}{(\lambda-1)^2\{\lambda^4-\mathrm{L}(1-\epsilon)\}}, \quad (8)$$

since $\qquad \mu(\mathrm{N}+1)^2=\xi-\eta=\epsilon\xi.$

In equation (8) Ψ becomes infinite when $\lambda=1$ and when $\lambda^4=\mathrm{L}(1-\epsilon)$.

The first of these assumptions is inconsistent with equation (7), and by substituting from $\lambda^4=\mathrm{L}(1-\epsilon)$ in (7) we get

$$(\lambda-1+\epsilon)^2=-\epsilon(1-\epsilon),$$

which is only possible if either ϵ or $1-\epsilon$ is negative, which they can never be, as ξ is always $>\eta$, and η and ξ are always positive.

In the case when the permeability is very great, $\xi=\eta=\mathrm{N}\mu^2$ nearly and ϵ may be neglected in equation (7), so that $\lambda^3=\mathrm{L}$.

If, however, $\lambda-1$ is very small, *i. e.* if the shells are very thin, the term in ϵ may be comparable with the others, and the approximation is not legitimate. In cases where it may be applied we get from equation (8) the very simple result

$$\Psi/\psi_0=\frac{(\mathrm{N}+1)^4}{\mathrm{N}^2\mu^2}\frac{\lambda^3}{(\lambda-1)^3}=\frac{\mathrm{L}\epsilon^2}{(\sqrt[3]{\mathrm{L}}-1)^3}.$$

In this case also the four quantities α_0, α_1, α_2, α_3 are in Geometrical Progression.

If $\alpha_1=\alpha_2$, $\alpha_0/\alpha_3=\lambda^2=\mathrm{L}$, so that (7) reduces to

$$(\sqrt{\mathrm{L}}-1)^2=\epsilon, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

of which the root greater than unity is $\sqrt{\mathrm{L}}=1+\sqrt{\epsilon}$.

This furnishes a limit below which the equations do not apply.

Below the limiting case lamination must be disadvantageous.

If we remove a thin slice either from the exterior or from the interior surface of a single shell, the shielding power is impaired; and if the shielding does not attain a maximum when this spherical crack is supposed to traverse the shell from inside to outside, it must always be less than if there is no crack.

Next, comparing the shielding effect of a single shell with that of a double shell of which the innermost and outermost radii are the same as that of the single shell, if we write in equation (2) α_3 for α_1, and put $\eta=\xi(1-\epsilon)$, we find by comparing with (4) that the single shell will shield best if

$$\frac{\alpha_0\epsilon}{\alpha_0-\alpha_3(1-\epsilon)}<\frac{\alpha_0\alpha_1\alpha_2\epsilon^2}{-(1-\epsilon)\alpha_2(\alpha_0-\alpha_1)(\alpha_2-\alpha_3)+\alpha_1\{\alpha_2-\alpha_3(1-\epsilon)\}\{\alpha_0-\alpha_1(1-\epsilon)\}},$$

which reduces to

$$\left(\frac{\alpha_2}{\alpha_3}-1\right)\left(\frac{\alpha_0}{\alpha_1}-1\right)<\epsilon.$$

Now if we diminish α_1, *i. e.* increase the outer radius of the inner shell, α_0/α_1 is increased; and in like manner by diminishing the inner radius of the outer shell α_2/α_3 is increased. Hence the largest value of the left-hand side of the inequality is attained when $\alpha_1=\alpha_2$, so that if $\alpha_2/\alpha_3=\lambda_1$, and $\alpha_0/\alpha_1=\lambda$, $\lambda\lambda_1=\mathrm{L}$.

Thus

$$(\lambda_1-1)(\lambda-1)=\left(\frac{L}{\lambda}-1\right)(\lambda-1),$$

of which the largest value is $(\sqrt{L}-1)^2$.

If ϵ is greater than $(\sqrt{L}-1)^2$, lamination is injurious, the limit thus fixed being the same as that previously found.

Above this limit, if for a given α_0 and α_3 we choose α_1 and α_2 so that

$$\left(\frac{\alpha_2}{\alpha_3}-1\right)\left(\frac{\alpha_0}{\alpha_1}-1\right)<\epsilon,$$

the two shells will shield better than one, *i. e.* better than if the whole of the space between the innermost and outermost surfaces were filled with the permeable material.

It of course follows *à fortiori* that the best arrangement of the two shells will, under these circumstances, give better results than the single shell, a conclusion which can be verified directly.

For we must by (8) and (2) have

$$\frac{L^2\epsilon^2}{(\lambda-1)^2\{\lambda^4-L(1-\epsilon)\}}<\frac{L\epsilon}{L-(1-\epsilon)},$$

where

$$\lambda^3(\lambda-1)-L(\lambda-1+\epsilon)=0.$$

Eliminating L, the inequality becomes

$$(1-\epsilon)\{\epsilon-(\lambda-1)^2\}^2>0.$$

which, since ϵ is always positive and <1, must always be true.

At the limiting point the curves which give the relations between L and the Ψ's for one and two shells respectively touch each other.

For, as in the case of a single shell,

$$\frac{\Psi_0}{\psi}=\frac{\epsilon L}{L-(1-\epsilon)},$$

$$\frac{1}{\psi_0}\frac{d\Psi}{dL}=-\frac{\epsilon(1-\epsilon)}{\{L-(1-\epsilon)\}^2}.$$

At the limiting point $L=(1+\sqrt{\epsilon})^2$;

$$\therefore\ \frac{1}{\psi_0}\frac{d\Psi}{dL}=-\frac{1-\sqrt{\epsilon}}{4(1+\sqrt{\epsilon})}.$$

In the case of two shells we have from (7),

$$\{\lambda^2(4\lambda-3)-\mathrm{L}\}\frac{d\lambda}{d\mathrm{L}}=\lambda-1+\epsilon,$$

and since at the limiting point $\lambda^2=\mathrm{L}=(1+\sqrt{\epsilon})^2$,

$$\frac{d\lambda}{d\mathrm{L}}=\frac{1}{4(1+\sqrt{\epsilon})}.$$

Also from (8),

$$\frac{1}{\psi_0}\frac{d\Psi}{d\mathrm{L}}=\frac{2\mathrm{L}\epsilon^2}{(\lambda-1)^2\{\lambda^4-\mathrm{L}(1-\epsilon)\}}+\frac{\mathrm{L}^2\epsilon^2(1-\epsilon)}{(\lambda-1)^2\{\lambda^4-\mathrm{L}(1-\epsilon)\}^2}$$
$$-2\left\{\frac{1}{\lambda-1}+\frac{2\lambda^3}{\lambda^4-\mathrm{L}(1-\epsilon)}\right\}\frac{\mathrm{L}^2\epsilon^2}{(\lambda-1)^2\{\lambda^4-\mathrm{L}(1-\epsilon)\}}\frac{d\lambda}{d\mathrm{L}};$$

and if we substitute in this the values of λ, L, and $d\lambda/d\mathrm{L}$ proper to the limiting point, we get

$$\frac{1}{\psi_0}\frac{d\Psi}{d\mathrm{L}}=-\frac{1-\sqrt{\epsilon}}{4(1+\sqrt{\epsilon})},$$

which is the same as the value obtained in the case of the single shell.

The shielding factor in the limiting case is obtained by putting $\lambda^2=\mathrm{L}=(1+\sqrt{\epsilon})^2$ in (8), or in the corresponding expression for a single shell. In either case

$$\frac{\Psi}{\psi_0}=\frac{\sqrt{\epsilon}(1+\sqrt{\epsilon})}{2}.$$

Since

$$\epsilon=\frac{\xi-\eta}{\xi}=\frac{\mu(\mathrm{N}+1)^2}{(\mathrm{N}\mu+1)(\mathrm{N}+\mu)},$$

which increases with N, *i. e.* with $(n+1)/n$, and therefore diminishes as n increases, the thickness of the shell at the limit is less for the higher harmonic terms, but the shielding at the limit is better.

If μ is so large, however, that we may write

$$\epsilon=\frac{(\mathrm{N}+1)^2}{\mathrm{N}\mu}=\frac{(2n+1)^2}{n(n+1)\mu},$$

the limiting thicknesses and the shielding factors for the principal harmonic terms vary very slowly. If $\mu = 400$ we get :—

n.	ϵ.	$L = (1+\sqrt{\epsilon})^2$.	$a_1/a_0 = (1+\sqrt{\epsilon})^{\frac{2}{3}}$.	Ψ/ψ_0.
1.	·0112	1·223	1·069	·058
2.	·0104	1·214	1·067	·056
3.	·0102	1·212	1·066	·055

Thus the limiting thickness in the case of the first three harmonic terms varies only from 6·9 per cent. to 6·6 per cent. of the radius, and the shielding factor from ·058 to ·055.

If there are three concentric shells, by differentiating the expression for Ψ in equation (3) we get

$$\left.\begin{aligned}
&\frac{D^2 d\Psi}{\psi_0\mu^3(N+1)^6\alpha_0\alpha_1\alpha_2\alpha_3\alpha_4} \\
&\quad = [\xi\eta\alpha_1\alpha_4(\alpha_4-\alpha_5)\{\eta\alpha_1(\alpha_2-\alpha_3)-\alpha_2(\eta\alpha_2-\xi\alpha_3)\} \\
&\qquad\qquad +\eta\alpha_1\alpha_3(\xi\alpha_4-\eta\alpha_5)\{\xi\alpha_2(\alpha_2-\alpha_3)-\alpha_1(\xi\alpha_2-\eta\alpha_3)\}]\,\frac{d\alpha_0}{\alpha_0} \\
&\qquad +[\eta\alpha_3(\xi\alpha_4-\eta\alpha_5)\{(\xi\alpha_2-\eta\alpha_3)\alpha_1{}^2-\xi\alpha_0\alpha_2(\alpha_2-\alpha_3)\} \\
&\qquad\qquad +\xi\eta\alpha_4(\alpha_4-\alpha_5)\{(\eta\alpha_2-\xi\alpha_3)\alpha_0\alpha_2-\eta\alpha_1{}^2(\alpha_2-\alpha_3)\}]\,\frac{d\alpha_1}{\alpha_1} \\
&\qquad +[\eta\alpha_3(\xi\alpha_4-\eta\alpha_5)\{-(\xi\alpha_0-\eta\alpha_1)\alpha_1\alpha_3+\xi\alpha_2{}^2(\alpha_0-\alpha_1)\} \\
&\qquad\qquad +\xi\eta\alpha_4(\alpha_4-\alpha_5)\{(\xi\alpha_0-\eta\alpha_1)\alpha_1\alpha_3-\eta\alpha_2{}^2(\alpha_0-\alpha_1)\}]\,\frac{d\alpha_2}{\alpha_2} \\
&\qquad +[\eta\alpha_3{}^2(\xi\alpha_4+\eta\alpha_5)\{\alpha_1(\xi\alpha_0-\eta\alpha_1)-\xi\alpha_2(\alpha_0-\alpha_1)\} \\
&\qquad\qquad +\xi\eta\alpha_2\alpha_4(\alpha_4-\alpha_5)\{\eta\alpha_2(\alpha_0-\alpha_1)-\alpha_1(\xi\alpha_0-\eta\alpha_1)\}]\,\frac{d\alpha_3}{\alpha_3} \\
&\qquad +[\eta\alpha_1(\xi\alpha_0-\eta\alpha_1)\{\xi\alpha_4{}^2(\alpha_2-\alpha_3)-\alpha_3\alpha_5(\xi\alpha_2-\eta\alpha_3)\} \\
&\qquad\qquad +\xi\eta\alpha_2(\alpha_0-\alpha_1)\{(\xi\alpha_3-\eta\alpha_2)\alpha_4{}^2+\eta\alpha_3\alpha_5(\alpha_2-\alpha_3)\}]\,\frac{d\alpha_4}{\alpha_4} \\
&\qquad +[\eta\alpha_1(\xi\alpha_0-\eta\alpha_1)\{\xi\alpha_4(\alpha_2-\alpha_3)+\alpha_3(\xi\alpha_2-\eta\alpha_3)\} \\
&\qquad\qquad +\xi\eta\alpha_2(\alpha_0-\alpha_1)\{(\eta\alpha_2-\xi\alpha_3)\alpha_4-\eta(\alpha_2-\alpha_3)\alpha_3\}]\,d\alpha_5.
\end{aligned}\right\} \quad (10)$$

We note that in this, as in the previous case, there can be no maximum or minimum if α_0 or α_5 are independent variables.

Further, writing the coefficient of $d\alpha_0$ in the form

$$\eta\alpha_1\alpha_4(\xi\alpha_4-\xi\alpha_5)\{\eta\alpha_1(\alpha_2-\alpha_3)-\alpha_2(\eta\alpha_2-\xi\alpha_3)\}$$
$$-\eta\alpha_1\alpha_3(\xi\alpha_4-\eta\alpha_5)\{\alpha_1(\xi\alpha_2-\eta\alpha_3)-\xi\alpha_2(\alpha_2-\alpha_3)\},$$

if we remember that the α's diminish from α_1 to α_5 and that ξ is $> \eta$, it is easy to see that the coefficient is the difference of two positive quantities of which the second is the larger.

For

$$\xi\alpha_4-\eta\alpha_5>\xi\alpha_4-\xi\alpha_5$$

and

$$\{\alpha_1(\xi\alpha_2-\eta\alpha_3)-\xi\alpha_2(\alpha_2-\alpha_3)-\{\eta\alpha_1(\alpha_2-\alpha_3)-\alpha_2(\eta\alpha_2-\xi\alpha_3)\}$$
$$=(\xi-\eta)\alpha_2(\alpha_1-\alpha_2),$$

which is positive.

Hence the whole coefficient is negative, or Ψ diminishes as α_0 increases, *i.e.* as the radius a_0 diminishes. Thus, as we should expect, we obtain the same result as before, viz., that the reduction of the radius of the innermost surface always improves the shielding. In like manner if a_5 increases the shielding improves.

Let us now suppose, as before, that the outermost and innermost radii are given. The best arrangement is then found by equating the coefficients of $d\alpha_1$, $d\alpha_2$, and $d\alpha_3$, and $d\alpha_4$ to zero.

By eliminating the terms in α_4 and α_5 from the first pair, and also from the second pair of these equations, we get

$$\alpha_2\alpha_3(\alpha_1-\alpha_2)\{\xi\alpha_0(\alpha_0-\alpha_1)\alpha_2-\alpha_1{}^2(\xi\alpha_0-\eta\alpha_1)\}=0 \quad . \quad (11)$$

and

$$\alpha_3(\alpha_2-\alpha_3)\{\xi\eta(\alpha_0-\alpha_1)^2\alpha_2{}^3+(\xi\alpha_0-\eta\alpha_1)^2\alpha_1{}^2\alpha_3$$
$$-\xi(\xi\alpha_0-\eta\alpha_1)\alpha_1\alpha_2(\alpha_2+\alpha_3)(\alpha_0-\alpha_1)=0 \quad . \quad (12)$$

In the cross multiplications by which these equations are obtained, we assume $\alpha_4-\alpha_5\neq 0$.

If $\alpha_1=\alpha_2$ or $\alpha_2=\alpha_3$ two of the shells are fused into one. Putting this case aside and substituting for $(\xi\alpha_0-\eta\alpha_1)/(\alpha_0-\alpha_1)$ in the second equation from the first we get

$$\alpha_1\alpha_2(\xi\alpha_0-\eta\alpha_1)-\xi\alpha_0\alpha_3(\alpha_0-\alpha_1)=0.$$

Hence and from (11),

$$\frac{\xi\alpha_0-\eta\alpha_1}{\xi\alpha_0(\alpha_0-\alpha_1)}=\frac{\alpha_3}{\alpha_1\alpha_2}=\frac{\alpha_2}{\alpha_1^2}, \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

$$\therefore \quad \alpha_3\alpha_1=\alpha_2^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

Again equating the coefficient of $d\alpha_3$ to zero, substituting for $(\xi\alpha_0-\eta\alpha_1)/(\alpha_0-\alpha_1)$, rearranging the terms, and substituting again, we get

$$\frac{\xi\alpha_4-\eta\alpha_5}{\xi\alpha_4(\alpha_4-\alpha_5)}=\frac{\alpha_0\alpha_2^2}{\alpha_1^2\alpha_3^2}=\frac{\alpha_0}{\alpha_2^2} \text{ by (14)}, . \quad . \qquad (15)$$

Next, equating the coefficient of $d\alpha_4$ to zero, we get

$$\frac{\xi\alpha_4^2-\eta\alpha_3\alpha_5}{\alpha_4^2-\alpha_3\alpha_5}=\frac{\alpha_2}{\alpha_3}\times\frac{\xi\alpha_0-\eta\alpha_1}{\alpha_0-\alpha_1}=\frac{\xi\alpha_0}{\alpha_1}.$$

This again reduces to

$$\frac{\alpha_4^2}{\alpha_3\alpha_5}=\frac{\xi\alpha_0-\eta\alpha_1}{\xi(\alpha_0-\alpha_1)}=\frac{\alpha_0\alpha_2}{\alpha_1^2} \text{ by (13)}, \quad . \quad . \qquad (16)$$

Again, since $\alpha_2^2=\alpha_3\alpha_1$,

$$\therefore \quad \alpha_3\alpha_1(\xi\alpha_4-\eta\alpha_5)=\xi\alpha_0\alpha_4(\alpha_4-\alpha_5). \quad . \quad . \quad . \quad . \qquad (17)$$

But

$$\alpha_3\alpha_5(\xi\alpha_0-\eta\alpha_1)=\xi\alpha_4^2(\alpha_0-\alpha_1),$$

or

$$\xi\alpha_0(\alpha_3\alpha_5-\alpha_4^2)=\alpha_1(\eta\alpha_3\alpha_5-\xi\alpha_4^2).$$

Multiplying this by (17) and putting $\alpha_2^2=\alpha_3\alpha_1$,

$$\alpha_3(\xi\alpha_4-\eta\alpha_5)(\alpha_3\alpha_5-\alpha_4^2)=\alpha_4(\alpha_4-\alpha_5)(\eta\alpha_3\alpha_5-\xi\alpha_4^2),$$

$$\therefore (\alpha_3-\alpha_4)\{\alpha_3\alpha_5(\xi\alpha_4-\eta\alpha_5)-\xi\alpha_4^2(\alpha_4-\alpha_5)\}=0.$$

If $\alpha_3=\alpha_4$ the problem is reduced to the case of two shells.

If $\alpha_3\neq\alpha_4$,

$$\frac{\xi\alpha_4-\eta\alpha_5}{\xi(\alpha_4-\alpha_5)}=\frac{\alpha_4^2}{\alpha_3\alpha_5}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

Hence the equations (13), (15), (16), and (18) finally reduce to

$$\frac{\xi\alpha_0-\eta\alpha_1}{\xi(\alpha_0-\alpha_1)}=\frac{\alpha_3\alpha_0}{\alpha_1\alpha_2}=\frac{\alpha_2\alpha_0}{\alpha_1^2}=\frac{\alpha_4^2}{\alpha_3\alpha_5}=\frac{\alpha_0\alpha_4}{\alpha_2^2}. \quad . \quad . \quad . \qquad (19)$$

It must be remembered that in obtaining these equations we have thrown out the alternative cases where any two consecutive values of α become identical; *i. e.* cases in which the three shells become two, either by fusion or by the dispearance of one.

Thus the solution for the case of two shells is contained in the general equations for three shells, but the equality of the above groups of ratios holds good only when there are three.

It remains to assure ourselves that in the case of three shells all the variables give minima simultaneously.

Thus, considering the coefficient of $d\alpha_1$ the coefficient of α_1^2 is

$$\begin{gathered}\eta\alpha_3(\xi\alpha_4-\eta\alpha_5)(\xi\alpha_2-\eta\alpha_3)\\ -\eta\alpha_4(\xi\alpha_4-\xi\alpha_5)(\eta\alpha_2-\eta\alpha_3).\end{gathered}$$

Comparing these term by term, we see that the terms are either equal or are less in the second expression. Hence the coefficient of α_1^2 is positive. If, then, α_1 is a little less than the critical value, the coefficient of $d\alpha_1$ must be negative. Hence, as α_1 increases up to the critical value Ψ diminishes, *i. e.* the critical value of α_1 corresponds to a minimum.

The other coefficients if treated in the same way lead to the same conclusion. We may now proceed to put equation (19) into the simplest form for calculation.

Since

$$\frac{\alpha_2}{\alpha_3}=\frac{\alpha_1}{\alpha_2},$$

$$\therefore\quad \frac{\alpha_4}{\alpha_3\alpha_5}=\frac{\alpha_0}{\alpha_2^{\,2}}=\frac{\alpha_0}{\alpha_1\alpha_3},\quad \therefore\ \frac{\alpha_4}{\alpha_5}=\frac{\alpha_0}{\alpha_1},$$

and since

$$\frac{\alpha_2\alpha_0}{\alpha_1^{\,2}}=\frac{\alpha_4^{\,2}}{\alpha_3\alpha_5},\quad \therefore\ \frac{\alpha_2}{\alpha_1}=\frac{\alpha_4}{\alpha_3};$$

so that we have three independent equations, viz.,

$$\left.\begin{aligned}&\frac{\alpha_0}{\alpha_1}=\frac{\alpha_4}{\alpha_5}=\lambda\text{ say,}\\ \text{and}\qquad&\frac{\alpha_1}{\alpha_2}=\frac{\alpha_2}{\alpha_3}=\frac{\alpha_3}{\alpha_4}=\lambda'\text{ say,}\end{aligned}\right\}\quad .\ .\ .\ .\quad (20)$$

together with

$$(\xi\lambda-\eta)\lambda'=\xi(\lambda-1)\lambda\,;\quad .\ .\ .\ .\ .\ .\quad (21)$$

also

$$\frac{\alpha_0}{\alpha_5}=\lambda^2\lambda'^3=\text{L say as before.}$$

So that equation (21) becomes

$$(\xi\lambda-\eta)\,L^{\frac{1}{3}}=\xi(\lambda-1)\lambda^{\frac{5}{3}}$$

or

$$(\lambda-1+\epsilon)\,L^{\frac{1}{3}}=(\lambda-1)\lambda^{\frac{5}{3}}. \quad . \quad . \quad . \quad . \qquad (22)$$

If $\lambda'=1$ all the shells fuse into one (without passing through the intermediate stage of two shells). In this case $\lambda^2=L$.

Substituting in (22) we get

$$(\sqrt{L}-1)^2=\epsilon,$$

which is the same as (9).

Thus the ratio of the outermost to the innermost radius below is the same for two and for three shells.

The maxima of advantage in the two cases are independent, and to determine between them their magnitudes must be compared.

Substituting in (3) from equation (20) and simplifying by means of (22), we finally get

$$\frac{\Psi}{\psi_0}=\frac{\epsilon^3L^{\frac{4}{3}}}{(\lambda-1)^4\{\lambda^{\frac{8}{3}}-L^{\frac{1}{3}}(1-\epsilon)\}}, \quad . \quad . \qquad (23)$$

where λ is given by equation (22).

In the limiting case when $\lambda^2=L=(1+\sqrt{\epsilon})^2$,

$$\Psi/\psi_0=\frac{\sqrt{\epsilon}(1+\sqrt{\epsilon})}{2},$$

which is, as it should be, the same as the value obtained by inserting the same values of λ and L in the corresponding expression for two shells. When ϵ is negligible as compared with λ and L we get from (22) $\lambda^5=L$, and from (23)

$$\frac{\Psi}{\psi_0}=\frac{\epsilon^3L}{(\sqrt[5]{L}-1)^5}=\frac{\epsilon^3\lambda^5}{(\lambda-1)^5}=\frac{(N+1)^6}{N^3\mu^3}\times\frac{L}{(\sqrt[5]{L}-1)^5}.$$

If we take the numbers before employed, viz. :—

$$\mu=501, \quad \alpha_0=8\alpha_3, \quad N=2,$$

we have

$$\lambda=\sqrt[5]{8}=1\cdot516\,;$$

$$\therefore \quad \frac{\Psi}{\psi_0}=\frac{3^6}{500^3}\times\left(\frac{1}{0\cdot516}\right)^5=0\cdot00016.$$

The results so far obtained in the special case to which numerical calculation has been applied may be summed up as follows, the arrangement being the best in each case :—

	Volume of Material used.	External Field.
Single shell	1·0	0·018
Two shells...............	5·0	0·0006
Three shells	4·8	0·00016
Single shell	7·0	0·0102

The great advantage of the lamination of the shielding material is here well exhibited.

As the external field varies slowly in the neighbourhood of the minimum, a considerable economy of material may be effected without much sacrifice of efficiency. Thus, in the case of two shells, we have seen that whereas to secure the minimum external field (0·0006 of the unshielded field) the volume of the material used must (in the case considered) be five times the internal volume, a practically identical result (0·0007) can be obtained by the use of a volume 3·66, while an efficiency four times greater can be obtained by a volume 4·8.

Best arrangement in the form of Two Shells separated by an Air-gap when the innermost radius and the volume of the material are given.

So far I have supposed that both the innermost and outermost radii are given.

It is evident that we may reduce the number of the independent variables, and I now proceed to discuss two problems which might occur in practice. The enunciation of the first of these is given above.

In neither case is the solution very simple, and I therefore confine myself to the case of two shells when the magnet is in the centre. In this case $\alpha_1 = 1/a_1^3$ &c., so that the condition that the volume shall be constant is

$$\frac{1}{\alpha_3} - \frac{1}{\alpha_2} + \frac{1}{\alpha_1} - \frac{1}{\alpha_0} = \frac{1}{\mathrm{R}} \text{ say.}$$

Hence

$$\frac{d\alpha_3}{\alpha_3^2}-\frac{d\alpha_2}{\alpha_2^2}+\frac{d\alpha_1}{\alpha_1^2}=0.$$

Substituting for $d\alpha_2$ in equation (5), and remembering that $d\alpha_0=0$, we get as the conditions for a critical value

$$\xi\alpha_2(\alpha_0-\alpha_1)\{\alpha_2^2-\alpha_3^2\}-\alpha_1\alpha_3(\alpha_2-\alpha_3)(\xi\alpha_0-\eta\alpha_1) = 0 \quad . \quad (24)$$

and

$$\begin{aligned}&\{-\xi\eta\alpha_1\alpha_0(\alpha_2-\alpha_3)+\eta\alpha_1^2(\xi\alpha_2-\eta\alpha_3)\}\alpha_1\\&\qquad+\{\xi\eta\alpha_2^2(\alpha_0-\alpha_1)-\eta\alpha_1\alpha_3(\xi\alpha_0-\eta\alpha_1)\}\alpha_2=0. \quad . \quad . \quad (25)\end{aligned}$$

From the first of these we get

$$(\xi\alpha_0-\eta\alpha_1)\alpha_1\alpha_3=\xi\alpha_2(\alpha_0-\alpha_1)(\alpha_2+\alpha_3). \quad . \quad . \quad . \quad (26)$$

Substituting for $\xi\alpha_0-\eta\alpha_1$ in (25),

$$(\xi\alpha_2-\eta\alpha_3)\alpha_1^3=\xi\alpha_2\{\alpha_0\alpha_1(\alpha_2-\alpha_3)+\alpha_2\alpha_3(\alpha_0-\alpha_1)\},$$

multiply by α_1 and subtract,

$$\begin{aligned}\therefore \quad \alpha_1^2(\alpha_0\alpha_3-\alpha_1\alpha_2) &= \alpha_2\{-\alpha_0\alpha_1(\alpha_2-\alpha_3)\\&\qquad+(\alpha_0-\alpha_1)(\alpha_1\alpha_2+\alpha_1\alpha_3-\alpha_2\alpha_3)\};\end{aligned}$$

$$\therefore \quad \alpha_1(\alpha_0\alpha_3-\alpha_1\alpha_2)=\alpha_2\alpha_3(\alpha_0-\alpha_1);$$

or

$$\alpha_0\alpha_3(\alpha_1-\alpha_2)=\alpha_1\alpha_2(\alpha_1-\alpha_3). \quad . \quad . \quad . \quad . \quad (27)$$

But

$$\frac{1}{\alpha_3}-\frac{1}{\alpha_2}+\frac{1}{\alpha_1}-\frac{1}{\alpha_0}=\frac{1}{R},$$

or

$$(\alpha_0-\alpha_3)\alpha_2\alpha_1-(\alpha_1-\alpha_2)\alpha_0\alpha_3=\frac{\alpha_0\alpha_1\alpha_2\alpha_3}{R};$$

$\therefore$ by (27),

$$\alpha_2\alpha_1\{\alpha_0-\alpha_3-(\alpha_1-\alpha_3)\}=\frac{\alpha_0\alpha_1\alpha_2\alpha_3}{R};$$

$$\therefore \quad \alpha_0-\alpha_1=\frac{\alpha_0\alpha_3}{R}. \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Hence

$$\alpha_1=\alpha_0\left(1-\frac{\alpha_3}{R}\right).$$

By combining this with (27),

$$\alpha_2\left\{\alpha_0\left(1-\frac{\alpha_3}{R}\right)^2+\frac{\alpha_3^2}{R}\right\}=\alpha_0\alpha_3\left(1-\frac{\alpha_3}{R}\right); \quad . \quad (29)$$

also from (26) and (28),

$$(\xi\alpha_0-\eta\alpha_1)\alpha_1 = \frac{\xi\alpha_2\alpha_0}{\mathrm{R}}(\alpha_2+\alpha_3),$$

and substituting for α_2 from (29), and dividing by $1-\frac{\alpha_3}{\mathrm{R}}$, this becomes

$$\left\{\xi-\eta+\frac{\eta\alpha_3}{\mathrm{R}}\right\}\left\{\alpha_0^2\left(1-\frac{\alpha_3}{\mathrm{R}}\right)^3+\frac{2\alpha_0\alpha_3^2}{\mathrm{R}}\left(1-\frac{\alpha_3}{\mathrm{R}}\right)\right\}$$
$$-\frac{\xi}{\mathrm{R}}\alpha_3^2\alpha_0\left\{2-\frac{\alpha_3}{\mathrm{R}}\right\}-\frac{\eta\alpha_3^4}{\mathrm{R}^2}=0. \quad . \quad . \quad . \quad (30)$$

The coefficient of α_3^4 in this expression is negative when $\alpha_3=-\infty$.

It is positive when $\alpha_3=0$, and negative when $\alpha_3=\infty$.

Hence the equation has a positive root. In order that the solution may be applicable, this root must be such that

$$\frac{1}{\alpha_3}-\frac{1}{\alpha_0}>\frac{1}{\mathrm{R}};$$

i. e. the space enclosed between the inner boundary of the inner shell and the outer boundary of the outer one must be greater than the volume of the material used. The change in the sign of the expression from positive to negative as α_3 increases must therefore take place when

$$\frac{1}{\alpha_3}>\frac{1}{\alpha_0}+\frac{1}{\mathrm{R}},$$

or the expression must be negative when

$$\frac{1}{\alpha_3}=\frac{1}{\alpha_0}+\frac{1}{\mathrm{R}}.$$

Now equation (30) may be thrown into the form

$$\left(\frac{\xi-\eta}{\alpha_3}+\frac{\eta}{\mathrm{R}}\right)\left\{\alpha_0^2\left(\frac{1}{\alpha_3}-\frac{1}{\mathrm{R}}\right)^3+\frac{2\alpha_0}{\mathrm{R}}\left(\frac{1}{\alpha_3}-\frac{1}{\mathrm{R}}\right)\right\}$$
$$-\frac{\xi}{\mathrm{R}}\frac{\alpha_0}{\alpha_3}\left(\frac{2}{\alpha_3}-\frac{1}{\mathrm{R}}\right)-\frac{\eta}{\mathrm{R}^2}=0;$$

and putting

$$\frac{1}{\alpha_3}-\frac{1}{\mathrm{R}}=\frac{1}{\alpha_0},$$

the left-hand side becomes

$$\xi\left(\frac{1}{\mathrm{R}}+\frac{1}{\alpha_0}\right)^2\left\{\frac{\xi-\eta}{\xi}-\frac{\alpha_0}{\mathrm{R}}\right\}.$$

Hence the condition that there shall be an arrangement which gives a minimum external field is that

$$\frac{\alpha_0}{\mathrm{R}}>\frac{\xi-\eta}{\xi},\quad i.e.\ >\epsilon,\quad i.e.\ >\frac{9\mu}{(2\mu+1)(2+\mu)}.$$

If we confine ourselves to the case of a small magnet in the centre of the shell: in the limiting case

$$\mathrm{L}=\frac{\alpha_0}{\alpha_3}=1+\frac{\alpha_0}{\mathrm{R}}=1+\epsilon,$$

as against $$\mathrm{L}=(1+\sqrt{\epsilon})^2,$$

obtained when the internal and external radii were given.

If μ is nearly equal to unity, this reduces to

$$\frac{1}{\mathrm{R}}>\frac{1}{\alpha_0};$$

i.e. there will be no "best arrangement" unless the volume of the shielding material exceeds the volume of the space enclosed. If μ is very large, the expression becomes

$$\frac{1}{\mathrm{R}}>\frac{1}{\alpha_0}\times\frac{9}{2\mu}.$$

Thus, if $\mu=500$, $$\frac{1}{\mathrm{R}}>\frac{0{\cdot}009}{\alpha_0}.$$

There will be no "best arrangement" unless the volume of the shielding material is greater than (in round numbers) one hundreth of the volume enclosed; *i.e.* unless it is more than sufficient to form a shell the thickness of which is $=0{\cdot}003$ of the radius.

In this case therefore, as in that in which the radius of the outer surface was given, if very thin shells are used, lamination is injurious. The higher the permeability the smaller is the lower limit of thickness above which lamination is useful. If the permeability is great, the limit is $\frac{3}{2\mu}\times$ radius. If we put

$$\frac{1}{\alpha_3}=\frac{1}{\mathrm{R}}+\frac{1}{\alpha_0}$$

in equation (29), we get $\alpha_2=\alpha_3$, *i.e.* the thickness of the external shell vanishes in the limiting case.

I now proceed to give some numerical examples of the application of these formulæ. Equation (30) is troublesome, but when μ is large so that ξ and η are nearly equal it may be simplified.

It then reduces to

$$\alpha_0^2\left(\frac{1}{\alpha_3}-\frac{1}{\mathrm{R}}\right)^3+\frac{2\alpha_0}{\mathrm{R}}\left(\frac{1}{\alpha_3}-\frac{1}{\mathrm{R}}\right)-\frac{\alpha_0}{\alpha_3}\left(\frac{2}{\alpha_3}-\frac{1}{\mathrm{R}}\right)-\frac{1}{\mathrm{R}}=0. \quad . \quad . \quad (31)$$

We have already seen that when the radius of the outer shell is twice the smallest radius, the best result is obtained when the volume of the shells is five times the volume of the space enclosed by the smaller shell.

The above equation may be used to determine whether a still better arrangement is possible with the same volume of material, if we suppose the radius of the external surface variable.

If $\alpha_0=1$, $1/\mathrm{R}=5$, and equation (31) is found to have a root such that

$$1/\alpha_3 = 9{\cdot}5 \text{ nearly.}$$

This gives

$$\alpha_3 = \sqrt[3]{9{\cdot}5} = 2{\cdot}118\,;$$

and, as in the previous calculation we assumed $\alpha_3=2$, it is evident that the two critical points nearly coincide.

In this case we get

$$\alpha_1 = \alpha_0(1-\alpha_3/\mathrm{R}) = 9/19,$$

from

$$\alpha_2 = 18/101.$$

The magnitude of the shielded field is practically the same as before.

It must be remembered that neither the equation in α_3 nor the ratio Ψ/ψ_0 are valid when ξ is put $=\eta$ if the shells are thin or (which amounts to the same thing) if R is large.

The best method of dealing with the problem is, then, as follows :—

Write

$$\xi = \eta(1+\epsilon),$$

where ϵ is so small that terms in ϵ may be neglected, unless divided by a small quantity.

Then equation (30) may be written

$$\left(1+\frac{\epsilon \mathrm{R}}{\alpha_3}\right)\left\{\alpha_0{}^2\left(\frac{1}{\alpha_3}-\frac{1}{\mathrm{R}}\right)^3+\frac{2\alpha_0}{\mathrm{R}}\left(\frac{1}{\alpha_3}-\frac{1}{\mathrm{R}}\right)\right\}$$
$$-\frac{\alpha_0}{\alpha_3}\left(\frac{2}{\alpha_3}-\frac{1}{\mathrm{R}}\right)(1+\epsilon)-\frac{1}{\mathrm{R}}=0.$$

The terms in ϵ may therefore be neglected unless R/α_3 is large.

As an example, let $\mathrm{R}=10$, *i.e.* let the volume of the metal be one tenth of the volume of the enclosure.

If then, as before, we take $\mu=501$, $\alpha_0=1$,

$$\therefore \quad \xi=\eta(1+0{\cdot}009) \quad \text{and} \quad \epsilon=0{\cdot}009.$$

If we neglect all the terms in ϵ we get

$$\frac{1}{\alpha_3}=2{\cdot}17.$$

Hence $$\epsilon\mathrm{R}/\alpha_3 = {\cdot}1953,$$

which cannot be neglected when compared with unity.

Taking this term into account,

$$1/\alpha_3 = 1{\cdot}875.$$

As the radii vary inversely as the cube roots of the α's, the two values of the external radius are 1·233 and 1·294. As the shielding varies slowly when near a minimum value, as the outer shell is displaced outward, it is probable that no very grave error would be introduced even in this case by the use of the approximate equation. The error would, however, increase rapidly as the volume of material employed diminished.

Using the more correct value we get

$$\alpha_1=\alpha_0\left(1-\frac{\alpha_3}{\mathrm{R}}\right)=1-0{\cdot}0533=0{\cdot}9467,$$

$$-\frac{1}{\alpha_2}=\frac{1}{\mathrm{R}}-\frac{1}{\alpha_3}-\frac{1}{\alpha_1}+\frac{1}{\alpha_0}=0{\cdot}1-1{\cdot}875-1{\cdot}056+1=-1{\cdot}831.$$

Hence we have for the best arrangement,

$$\alpha_3=0{\cdot}5333, \quad \alpha_2=0{\cdot}5461, \quad \alpha_1=0{\cdot}9467, \quad \alpha_0=1,$$

or $$\alpha_3=1{\cdot}233, \quad \alpha_2=1{\cdot}223, \quad \alpha_1=1{\cdot}018, \quad \alpha_0=1.$$

It will be convenient to throw equation (4) into a form suitable for calculation when the shells are thin and the permeability is great.

With the same notation as that just employed it becomes

$$\Psi(\alpha_0-\alpha_1)(\alpha_2-\alpha_3)\left\{-\alpha_2+\alpha_1\left(1+\frac{\alpha_3\epsilon}{\alpha_2-\alpha_3}\right)\left(1+\frac{\alpha_1\epsilon}{\alpha_0-\alpha_1}\right)\right\}$$
$$=\frac{\psi_0\alpha_0\alpha_1\alpha_2(\mathrm{N}+1)^4}{\mathrm{N}^2\mu^2},$$

which, in the case under consideration, leads to the result

$$\Psi/\psi_0=0{\cdot}064.$$

By (2), if the same amount of material were concentrated in a single shell, we should have

$$\Psi/\psi_0=0{\cdot}089.$$

In this case therefore the advantage of the best arrangement is much less than if a larger amount of material were used.

No very great disadvantage is entailed in such a case if the outer shell be outside its best position, as, if we take $1/\alpha_3=3$, we get

$$\Psi/\psi_0=0{\cdot}069,$$

which does not differ very much from the minimum value.

Case of Two Shells when the largest and smallest radii and the volume of the material used are given.

In this case $d\alpha_0=d\alpha_3=0$, and as before

$$\frac{1}{\alpha_3}-\frac{1}{\alpha_2}+\frac{1}{\alpha_1}-\frac{1}{\alpha_0}=\frac{1}{\mathrm{R}},$$
$$\frac{1}{\alpha_2}-\frac{1}{\alpha_1}=\frac{1}{\rho},$$

where, since α_3 and α_0 are given, ρ is a known quantity.

Hence substituting from

$$\frac{d\alpha_2}{\alpha_2{}^2}=\frac{d\alpha_1}{\alpha_1{}^2}\text{ in (5)}$$

we get

$$[\xi\eta\{\alpha_1\alpha_2(\alpha_1+\alpha_2)-\alpha_0\alpha_2{}^2\}-\eta^2\alpha_1{}^2\alpha_3]\,(\alpha_2-\alpha_3)=0,$$

or

$$\alpha_2{}^2(\alpha_0-\alpha_1)-\alpha_2\alpha_1{}^2=-\frac{\eta}{\xi}\alpha_1{}^2\alpha_3.$$

But
$$\frac{1}{\alpha_2} = \frac{\rho+\alpha_1}{\rho\alpha_1},$$

$$\therefore \quad \frac{\rho^2(\alpha_0-\alpha_1)}{(\rho+\alpha_1)^2} - \frac{\rho\alpha_1}{\rho+\alpha_1} = -\frac{\eta\alpha_3}{\xi},$$

or
$$(\rho+\alpha_1)^2 = \frac{\xi\rho^2(\alpha_0+\rho)}{\xi\rho-\eta\alpha_3},$$

whence α_1 is known.

In the case when μ is so great that ξ may be taken $=\eta$, this becomes

$$(\rho+\alpha_1)^2(\rho-\alpha_3) = \rho^2(\alpha_0+\rho).$$

If the given volume is very nearly $=\frac{4}{3}\pi(a_3{}^3-a_0{}^3)$, $1/\rho$ is small.

In this case the solution reduces to

$$2\alpha_1\xi - \eta\alpha_3 = \xi\alpha_0,$$

or
$$\alpha_1 = \frac{\alpha_3\eta+\xi\alpha_0}{2\xi}.$$

If μ is large, this becomes

$$\alpha_1 = \frac{\alpha_3+\alpha_0}{2} \qquad \text{or} \qquad \frac{1}{a_1{}^3} = \frac{1}{2}\left(\frac{1}{a_3{}^2}+\frac{1}{a_0{}^2}\right).$$

Hence, if the external and internal radii are given, the best position for a narrow crack within the shell is such that the volume it encloses is the harmonic mean of the volumes enclosed by the inner and outer surfaces of the shell respectively.

Case of Two contiguous Shells of different Permeabilities.

It is evident that the question as to the best arrangement of two shells of different permeabilities cannot be very definitely answered.

If the dimensions of the shells are variable, the more of the more permeable material we use, the more complete will be the shielding.

If, however, the dimensions of the shells are fixed, under certain circumstances (such as when its thickness is much the greater) it will be better to make the outer shell of the more permeable material, while under other conditions the reverse arrangement may be best.

Though the problem is not of any great practical interest, it may be worth while to indicate the mode of solution.

Let $\mu_0=\mu_3=1$, and let the permeabilities of the two shells be μ and μ_2. Then we get

$$\begin{aligned}\Psi\{&-\alpha_1\alpha_2\mathrm{N}(\mathrm{N}\mu_1+\mu_2)(\mu_2-1)(\mu_1-1)-\alpha_1{}^2\mathrm{N}(\mathrm{N}+\mu_2)(\mu_1-\mu_2)(\mu_1-1)\\&+\alpha_0\alpha_2(\mathrm{N}\mu_1+1)\mathrm{N}(\mu_2-1)(\mu_1-\mu_2)+\alpha_0\alpha_1(\mathrm{N}\mu_1+1)(\mathrm{N}+\mu_2)(\mathrm{N}\mu_2+\mu_1)\\&=\psi_0\alpha_0\alpha_1(\mathrm{N}+1)^3\mu_1\mu_2.\end{aligned}$$

Hence, if Ψ_2 and Ψ_1 be the external potential when the permeability of the external shell is μ_2 and μ_1 respectively, we get, by exchanging the positions of the symbols μ_1 and μ_2 in the above expression and subtracting,

$$\begin{aligned}\psi\alpha_0\alpha_1(\mathrm{N}+1)^3\mu_1\mu_2\left\{\frac{1}{\Psi_2}-\frac{1}{\Psi_1}\right\}\\=(\mu_1-\mu_2)[&-\alpha_1{}^2\mathrm{N}\{2\mu_1\mu_2+(\mathrm{N}-1)(\mu_1+\mu_2)-2\mathrm{N}\}\\&-\alpha_1(\alpha_2+\alpha_0)\mathrm{N}(\mathrm{N}-1)(\mu_2-1)(\mu_1-1)\\&+\alpha_0\alpha_2\mathrm{N}\{2\mathrm{N}\mu_1\mu_2-(\mathrm{N}-1)(\mu_1+\mu_2)-2\}].\end{aligned}$$

Hence the exchange of materials causes no difference if either $\mu_1=\mu_2$ or if the second factor on the right-hand side vanishes.

If both the shells are thin let $a_2=a_1+t_1$, and $a_0=a_1-t_0$, where the squares of t_1 and t_2 may be neglected.

Then

$$\alpha_2=(a_1+t_2)^{-(2n+1)}=\alpha_1\left\{1-(2n+1)\frac{t_2}{a_1}\right\}$$

$$\alpha_0=(a_1-t_0)^{-(2n+1)}=\alpha_1\left\{1+(2n+1)\frac{t_0}{a_1}\right\}.$$

Substituting these approximate values, the second factor becomes

$$-(2n+1)\mathrm{N}(\mathrm{N}+1)(\mu_1\mu_2-1)(t_2-t_0)/a_1,$$

which proves that, as we should expect, if the shells are thin the order of their arrangement is indifferent.

Case of Three Shells of different Permeabilities.

The general relation between Ψ and ψ_0 when an internal space of permeability μ_0 is surrounded by three concentric spherical shells of permeabilities μ_1, μ_2, μ_3, and when the permeability of external space is μ_4, is

$$\begin{aligned}
&\psi_0\alpha_0\alpha_1\alpha_2 \,.\, \mu_0\mu_1\mu_2\mu_3(\mathrm{N}+1)^4/\Psi \\
&\quad =\alpha_1\alpha_2\alpha_3\mathrm{N}(\mu_3-\mu_4)(\mu_0-\mu_1)(\mathrm{N}\mu_2+\mu_3)(\mathrm{N}\mu_1+\mu_2) \\
&\qquad +\alpha_1\alpha_2^2\mathrm{N}(\mathrm{N}\mu_4+\mu_3)(\mu_2-\mu_3)(\mu_0-\mu_1)(\mathrm{N}\mu_1+\mu_2) \\
&\qquad +\alpha_1^2\alpha_3\mathrm{N}^2(\mu_3-\mu_4)(\mu_2-\mu_3)(\mu_1-\mu_2)(\mu_0-\mu_1) \\
&\qquad +\alpha_1^2\alpha_2\mathrm{N}(\mathrm{N}\mu_4+\mu_3)(\mathrm{N}\mu_3+\mu_2)(\mu_1-\mu_2)(\mu_0-\mu_1) \\
&\qquad +\alpha_0\alpha_2\alpha_3\mathrm{N}(\mathrm{N}\mu_2+\mu_3)(\mathrm{N}\mu_1+\mu_0)(\mu_3-\mu_4)(\mu_1-\mu_2) \\
&\qquad +\alpha_0\alpha_2^2\mathrm{N}(\mathrm{N}\mu_4+\mu_3)(\mathrm{N}\mu_1+\mu_0)(\mu_2-\mu_3)(\mu_1-\mu_2) \\
&\qquad +\alpha_0\alpha_1\alpha_3\mathrm{N}(\mathrm{N}\mu_2+\mu_1)(\mathrm{N}\mu_1+\mu_0)(\mu_3-\mu_4)(\mu_2-\mu_3) \\
&\qquad +\alpha_0\alpha_1\alpha_2(\mathrm{N}\mu_1+\mu_0)(\mathrm{N}\mu_2+\mu_1)(\mathrm{N}\mu_3+\mu_2)(\mathrm{N}\mu_4+\mu_3).
\end{aligned}$$

If now we put $\mu_0=\mu_2=\mu_4=1$, this reduces to the case of two concentric shells of permeabilities μ_1 and μ_3 respectively in air and separated by an air-gap. The equation then becomes

$$\begin{aligned}
&\psi_0\alpha_0\alpha_1\alpha_2 \,.\, \mu_1\mu_3(\mathrm{N}+1)^4/\Psi \\
&\quad =-\alpha_1\alpha_2\alpha_3\mathrm{N}(\mu_3-1)(\mu_1-1)(\mathrm{N}+\mu_3)(\mathrm{N}\mu_1+1) \\
&\qquad +\alpha_1\alpha_2^2\mathrm{N}(\mathrm{N}+\mu_3)(\mu_3-1)(\mu_1-1)(\mathrm{N}\mu_1+1) \\
&\qquad +\alpha_1^2\alpha_3\mathrm{N}^2(\mu_3-1)^2(\mu_1-1)^2 \\
&\qquad -\alpha_1^2\alpha_2\mathrm{N}(\mathrm{N}+\mu_3)(\mathrm{N}\mu_3+1)(\mu_1-1)^2 \\
&\qquad +\alpha_0\alpha_2\alpha_3\mathrm{N}(\mathrm{N}+\mu_3)(\mathrm{N}\mu_1+1)(\mu_3-1)(\mu_1-1) \\
&\qquad -\alpha_0\alpha_2^2\mathrm{N}(\mathrm{N}+\mu_3)(\mathrm{N}\mu_1+1)(\mu_3-1)(\mu_1-1) \\
&\qquad -\alpha_0\alpha_1\alpha_3\mathrm{N}(\mathrm{N}+\mu_1)(\mathrm{N}\mu_1+1)(\mu_3-1)^2 \\
&\qquad +\alpha_0\alpha_1\alpha_2(\mathrm{N}\mu_1+1)(\mathrm{N}+\mu_1)(\mathrm{N}\mu_3+1)(\mathrm{N}+\mu_3).
\end{aligned}$$

If we write

$$\begin{aligned}
&\xi_1=(\mathrm{N}\mu_1+1)(\mathrm{N}+\mu_1), \quad && \xi_3=(\mathrm{N}\mu_3+1)(\mathrm{N}+\mu_3), \\
&\eta_1=\mathrm{N}(\mu_1-1)^2, && \eta_3=\mathrm{N}(\mu_3-1)^2, \\
&\mathrm{P}=\mathrm{N}(\mu_3-1)(\mu_1-1)(\mathrm{N}+\mu_3)(\mathrm{N}\mu_1+1), &&
\end{aligned}$$

the equation reduces to

$$\begin{aligned}
&\psi_0\alpha_0\alpha_1\alpha_2\mu_1\mu_3(\mathrm{N}+1)^4\Psi \\
&=\alpha_1(\xi_3\alpha_2-\eta_3\alpha_3)(\xi_1\alpha_0-\eta_1\alpha_1)-\mathrm{P}\alpha_2(\alpha_0-\alpha_1)(\alpha_2-\alpha_3),
\end{aligned}$$

which, if we proceed to put $\mu_1=\mu_3$, further reduces to the expression previously obtained.

Differentiating,

$$\left.\begin{aligned}
&\frac{D^2 d\Psi}{\psi_0\mu_1\mu_3(N+1)^4}\\
&\quad=\{P\alpha_2\alpha_1(\alpha_2-\alpha_3)-\eta\alpha_1{}^2(\xi_3\alpha_2-\eta_3\alpha_3)\}\alpha_1\alpha_2 d\alpha_0\\
&\qquad+\{-P\alpha_2\alpha_0(\alpha_2-\alpha_3)+\eta_1\alpha_1{}^2(\xi_3\alpha_2-\eta_3\alpha_3)\}\alpha_0\alpha_2 d\alpha_1\\
&\qquad+\{P\alpha_2{}^2(\alpha_0-\alpha_1)-\eta_3\alpha_1\alpha_3(\xi_1\alpha_0-\eta_1\alpha_1)\}\alpha_0\alpha_1 d\alpha_2\\
&\qquad+\{-P\alpha_2(\alpha_0-\alpha_1)+\eta\alpha_1(\xi_1\alpha_0-\eta_1\alpha_1)\}\alpha_0\alpha_1\alpha_2 d\alpha_3.
\end{aligned}\right\}\ (32)$$

If the inner shell and the external radius are given, the best value of $\alpha_2{}^2$ is deduced from the equation

$$\alpha_2{}^2=\frac{\eta_3}{P}\cdot\frac{\alpha_1\alpha_3}{\alpha_0-\alpha_1}(\xi_1\alpha_0-\eta_1\alpha_1).$$

Comparing this with equation (6) we see that the new value of α_2 is greater than that obtained when the permeability of the external shell is the same as that of the inner one if

$$\eta_3/P > 1/\xi_1\,;$$

$$\textit{i. e.,}\quad \text{if } (\mu_3-1)(N+\mu_1) > (\mu_1-1)(N+\mu_3),$$

$$\textit{i. e.,}\quad \text{if } \mu_3 > \mu_1.$$

In like manner, if the outer shell and the internal radius of the inner shell are given, the best arrangement requires that

$$\alpha_1{}^2=\frac{P\alpha_2\alpha_0(\alpha_2-\alpha_3)}{\eta_1(\xi_3\alpha_2-\eta_3\alpha_3)},$$

which is greater than the corresponding value when the permeabilities of the shells are the same as that of the inner shell if

$$\frac{P}{\eta_1(\xi_3\alpha_2-\eta_3\alpha_3)} > \frac{\xi_1}{\xi_1\alpha_2-\eta_1\alpha_3},$$

$$\textit{i. e.,}\quad \text{if } \alpha_2\xi_1(P-\eta_1\xi_3)\quad \alpha_3\eta_1(P-\xi_1\eta_3).$$

But

$$P-\eta_1\xi_3=N(\mu_1-1)(N+\mu_3)(N+1)(\mu_3-\mu_1),$$
$$P-\xi_1\eta_3=N(\mu_3-1)(N\mu_1+1)(N+1)(\mu_1-\mu_3)\,;$$

and therefore the two sides of the inequality are always of opposite signs.

Hence, if the permeability of the outer shell is increased, the values of α_2 and α_1 corresponding to the best arrangement under the circumstances supposed increase, *i. e.*, the outer shell increases and the inner shell diminishes in thickness.

In the case, therefore, of material such as iron, if the field within the outer shell is such that the permeability of the outer shell is the greater (which will generally be the case when the enclosed space is protected), it will be better to make the external shell relatively thicker than the above calculation would indicate. Conversely, if the inner shell has the higher permeability its thickness should be increased.

There is no difficulty in eliminating α_1 and α_2 in turn from the relations obtained by equating the coefficients of $d\alpha_1$ and $d\alpha_2$ in (32) to zero, and thus finding the best values in any given case for which the equations can be solved by trial.

Summary.

In conclusion it is perhaps desirable to summarize the principal results of the foregoing discussion. In doing so I shall confine myself exclusively to the corresponding cases of a small magnet placed in the centre of the shells, and of shells placed in a uniform field, and thus the interpretation of the symbols is valid for these cases only.

The results depend upon two quantities, viz., the ratio of the outermost to the innermost radius of the series of shells. In the paper the cube of this quantity is indicated by the symbol $L=\alpha_0/\alpha_n=a_n{}^3/a_0{}^3$, where n refers to the outermost surface.

The other quantity ϵ is a function of the permeability. In the cases which I am now considering,

$$\epsilon=\frac{9\mu}{(2\mu+1)(\mu+2)}=\frac{9}{2\mu}\text{ approximately,}$$

when μ is large.

If we suppose the innermost radius of the shells (a_0) to be fixed, and the thickness of the space utilized for shielding to increase gradually from zero to a thickness t, the material employed is most efficient if fashioned in the form of a single shell until

$$\frac{t}{a_0}=(1+\epsilon)^{\frac{1}{3}}-1=\frac{3}{2\mu}\text{ approximately,}$$

if μ is large.

After that the material will be better employed if we divide it into separated shells; but the result obtained by the best use of the given quantity of material can be still further improved by filling up the space between the shells with more of the material, provided the thickness of the whole shell occupied by the shielding machinery is less than the value given by

$$\frac{t}{a_0}=(1+\sqrt{\epsilon})^{\frac{2}{3}}-1=\sqrt{\frac{2}{\mu}},$$

if μ is large.

After this limit is passed it becomes positively injurious to fill the whole of the shielding space with the permeable material.

Better results can always be obtained by two shells with the same internal and external radii.

If the ratio of these radii ($a_3/a_0=\sqrt[3]{L}$) is given, the best arrangement of two shells is defined by the equations

$$\frac{a_3}{a_2}=\frac{a_1}{a_0}=\lambda^{\frac{1}{3}},$$

where λ is given by the equation

$$\lambda^3(\lambda-1)-L(\lambda-1+\epsilon)=0.$$

The best arrangement for three shells, if the ratio of the innermost and outermost radii is given ($a_5/a_0=\sqrt[3]{L}$), is defined by the equations:

$$\frac{a_5}{a_4}=\frac{a_1}{a_0}=\lambda^{\frac{1}{3}},$$

$$\frac{a_4}{a_3}=\frac{a_3}{a_2}=\frac{a_2}{a_1}=L^{\frac{1}{9}}\lambda^{-\frac{2}{9}},$$

where λ is defined by the equation

$$\lambda^{\frac{5}{3}}(\lambda-1)-L^{\frac{1}{3}}(\lambda-1+\epsilon)=0.$$

In both cases, when the permeability is great the best arrangement is that in which the radii of the successive bounding surfaces of the shells are in Geometrical Progression.

The ratio of the shielded to the unshielded field when the best arrangement is adopted is

$$\frac{L^2\epsilon^2}{(\lambda-1)^2\{\lambda^4-L(1-\epsilon)\}} \quad \text{and} \quad \frac{L^{\frac{4}{3}}\epsilon^3}{(\lambda-1)^4\{\lambda^{\frac{8}{3}}-L^{\frac{1}{3}}(1-\epsilon)\}}$$

for two and three shells respectively, λ being in each case determined from the equation given above as appropriate to that case.

When the permeability is great and the shells are not very thin, these expressions become

$$\frac{L\epsilon^2}{(\sqrt[3]{L}-1)^3} \quad \text{and} \quad \frac{L\epsilon^3}{(\sqrt[5]{L}-1)^5} \text{ respectively.}$$

The conditions for the best arrangement in the form of two shells, when (1) the innermost radius and the total volume of the permeable material, and (2) the innermost and outermost radii and the total volume of permeable material are given, are less symmetrical but are set forth above.

In any case of an arrangement in the form of two shells of materials of different permeability, in order to obtain the best result, the thickness of the shell formed of the more permeable material should be greater, and that of the shell of less permeable material should be less than is given by the above rules.

The equations are obtained by which the best arrangement can be calculated, if the permeabilities and external and internal radii are given.

DISCUSSION.

Prof. Minchin said the mathematical results were very simply expressed. Although the work was apparently restricted to zonal spherical harmonics, some of the important formulæ apply equally to general spherical harmonics. Referring to the difficulty of shielding by single thick shells, he pointed out that the equation giving the relation between the shielded and the unshielded fields with different thicknesses of shell represented a hyperbola with its asymptotes parallel to the axes; hence the shielding tended to a definite limit as the thickness increased indefinitely.

Mr. Evershed said he had been engaged for the last two years on the subject of magnetic shielding with a view to screening measuring instruments from external fields. In such cases it was not possible to use closed shells, and this introduced trouble. The best result he had yet obtained was to reduce the disturbance to about one-fifth. Another difficulty was introduced by the fact of the shield being magnetized by the current passing through the coil, and owing to hysteresis, the permeability was different according as the magnetization increased or decreased. By using an outer iron shell a great improvement had been effected. To obtain the best results it was important to have no joints in the shields. A coil frame with two shields of bent iron was exhibited.

Mr. J. Swinburne remarked that the subject divided itself into two: shielding instruments and shielding sources. If a dynamo itself be shielded, this did not prevent the currents in the leads producing magnetic disturbances. This was very important in ships. By using an alternator with revolving fields all disturbances could be avoided.

Dr. C. V. Burton enquired whether, by considering the hydrodynamical analogue of a porous material, the case of perforated shells could be elucidated.

Mr. A. P. Trotter wished to know if the homogeneity of the shield was of much consequence. At Oxford it had been found that a screen of four inches of scrap iron was better than boiler-plate.

Mr. Blakesley asked if the effect of moving a magnet sideways in a sphere had been observed. He thought the mathematics developed in the paper would be useful in working out the magnetic theory of the earth.

Prof. S. P. Thompson thought that taking the permeability as constant would not be quite correct, for μ was a function of the magnetization. Hence in the cases considered the outer shell would be the more permeable.

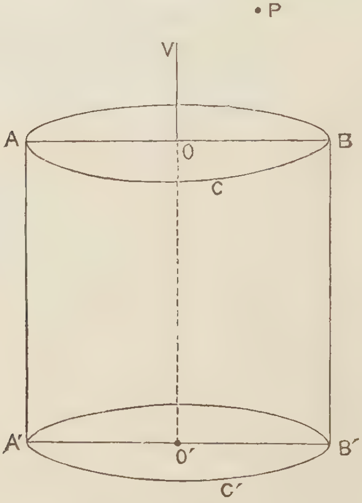
In his reply the President said scrap iron in contact was not like clear space, for there were comparatively free paths for the induction at the points of contact. As regards the shielding of the dynamo at Greenwich, Mr. Christie had written to say that the credit was due to the makers of the machine and shields, Messrs. Johnson and Phillips.

XXXIV. *Calculation of the Magnetic Field of a Current running in a Cylindrical Coil.* *By* Professor G. M. Minchin, *M.A.**

Let there be a series of very close circular currents running in the same sense and lying on a right cylinder of radius a whose axis is OO′ (fig. 1), and let it be required to find the magnetic potential of this system at any point, P, in space.

Replace each of these circular currents by its equivalent magnetic shell, which we shall take as a uniform circular plate coinciding with the aperture of the circle. Supposing the currents to circulate in the sense ACB, the upper surface of each plate (as seen in the figure) will be positive and the lower negative. Each circle being touched all round by the one below it, the negative surface of any plate will coincide with the positive of the one next below it; so that we shall be left with a terminal positive plate, ACB, and a terminal negative plate, A′C′B′, as the approximate equivalent of the

Fig. 1.

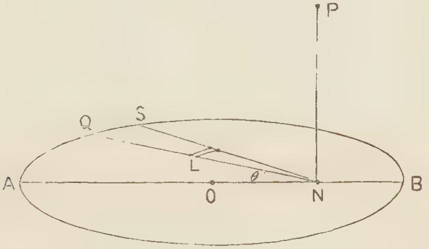


---

* Read December 8, 1893.

given coil of wire. We shall now calculate the potential produced at any point, P, by a uniform thin plate represented by AQB (fig. 2), the surface-density of attracting matter on this plate being m.

Fig. 2.



Of course it is well known that this potential can be exhibited by a series of Zonal Harmonics proceeding by powers, positive or negative, of the ratio $\frac{\mathrm{OP}}{a}$, where O is the centre of the plate and a its radius, according as P is near O or distant from it ; but when OP is not *very* much greater or *very* much less than a, the series is inconvenient, owing to the enormous number of terms that have to be taken to give a good approximation.

Hence we shall not use Spherical Harmonics.

Let PN be the perpendicular on the plate from P ; let AB be the diameter in which it is cut by the plane through the axis of the plate and P ; let Q, S be two very close points on the circumference, NQ making the angle θ with NA. Then we shall suppose the plate broken up into triangular strips such as QNS, and calculate the potential of each strip at P.

Let L be any point on NQ, let $\mathrm{NL}=\xi$; then the potential of the element $m\xi d\xi d\theta$ at P is $\frac{m\xi d\xi d\theta}{\mathrm{LP}}$, or $md\theta\frac{\xi d\xi}{\sqrt{z^2+\xi^2}}$, where $\mathrm{PN}=z$. If $\mathrm{NQ}=r$ we find by integration that the potential produced by the strip QNS is

$$md\theta(\sqrt{r^2+z^2}-z), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the potential, V, of the whole plate is

$$2m\int(\sqrt{r^2+z^2}-z)d\theta. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now if, as in the figure, the point N falls within the plate, the limits of θ are 0 and π; if N falls on the edge of the plate, at B, the limits are 0 and $\frac{\pi}{2}$; and if it falls outside the plate, the limits are 0 and 0. Taking θ as the independent variable would, then, give us three different expressions for V, according to the position of N; and hence we must choose a more convenient variable than θ. Let ϕ be the angle QOA, and change the expression (2) into one in which ϕ is the independent variable. We shall then have

$$r^2 = a^2 + x^2 + 2ax \cos\phi\,; \quad \tan\theta = \frac{a \sin\phi}{x + a\cos\phi}\,;$$

so that

$$\mathrm{V} = 2ma \int_0^\pi \frac{\sqrt{a^2+x^2+z^2+2ax\cos\phi} - z}{a^2+x^2+2ax\cos\phi}(a + x\cos\phi)\,d\phi\,;$$

or if we put $\mathrm{D} = a^2 + x^2 + 2ax\cos\phi$,

$$\frac{\mathrm{V}}{m} = \int_0^\pi \frac{\sqrt{z^2+\mathrm{D}} - z}{\mathrm{D}}(\mathrm{D} + a^2 - x^2)\,d\phi$$

$$= \int_0^\pi \sqrt{z^2+\mathrm{D}}\,d\phi + (a^2 - x^2)\int_0^\pi \frac{\sqrt{z^2+\mathrm{D}}}{\mathrm{D}}\,d\phi - z\int_0^\pi \left(1 + \frac{a^2-x^2}{\mathrm{D}}\right)d\phi.$$

Now

$$(a^2 - x^2)\int_0^\pi \frac{d\phi}{\mathrm{D}} = \pi\,;$$

so that

$$\frac{\mathrm{V}}{m} = -2\pi z + \int_0^\pi \sqrt{z^2+\mathrm{D}}\,d\phi + (a^2-x^2)\int_0^\pi \frac{\sqrt{z^2+\mathrm{D}}}{\mathrm{D}}\,d\phi$$

$$= -2\pi z + \int_0^\pi \sqrt{z^2+\mathrm{D}}\,d\phi + (a^2-x^2)\int_0^\pi \frac{d\phi}{\sqrt{z^2+\mathrm{D}}}$$

$$+ (a^2-x^2)z^2 \int_0^\pi \frac{d\phi}{\mathrm{D}\sqrt{z^2+\mathrm{D}}}. \quad (3)$$

Again, if we put $\phi = 2\omega$, we have

$$\mathrm{D} = (a+x)^2 - 4ax\sin^2\omega\,;$$

and if the distances PA, PB are denoted by ρ, ρ', respectively, we have

$$4ax = \rho^2 - \rho'^2,$$

so that

$$z^2 + \mathrm{D} = \rho^2 - (\rho^2 - \rho'^2)\sin^2\omega.$$

Let

$$k^2 = 1 - \frac{\rho'^2}{\rho^2}\,; \quad k'^2 = \frac{\rho'^2}{\rho^2}\,; \quad \Delta\omega = \sqrt{1 - k^2 \sin^2\omega}. \quad . \ (4)$$

Then (3) becomes

$$\frac{\mathrm{V}}{m} = -2\pi z + 2\rho \mathrm{E} + \frac{2}{\rho}(a^2 - x^2)\mathrm{K} + 2\frac{z^2}{\rho}\,\frac{a-x}{a+x}\,.\int_0^{\frac{\pi}{2}} \frac{1}{1 - \dfrac{4ax}{(a+x)^2}\sin^2\omega}\,.\,\frac{d\omega}{\Delta\omega}, \quad (5)$$

where E and K are the complete elliptic integrals of the second and first kinds with modulus k.

The integral in (5) is the complete elliptic integral of the third kind with modulus k and parameter $-\dfrac{4ax}{(a+x)^2}$. This parameter is numerically greater than the modulus; and we shall find it convenient to convert the integral into one in which the parameter is less than the modulus by the well-known rule that a function with parameter n can be converted into one with parameter $\dfrac{k^2}{n}$. If the angles PBA and PAB are denoted by θ and θ', respectively, we see that n, the parameter in (5), is $-\dfrac{k^2}{\cos^2\theta'}$; so that the new parameter will be simply $-\cos^2\theta'$.

Now we have the general result that

$$\Pi(n, \epsilon) + \Pi\left(\frac{k^2}{n}, \epsilon\right) = \frac{1}{\sqrt{\alpha}} \tan^{-1}\left(\frac{\sqrt{\alpha}\tan\epsilon}{\Delta\epsilon}\right) + \mathrm{K}(\epsilon),$$

where ϵ is the amplitude of each of the two functions of the third kind (denoted by Π), and $\alpha = (1+n)\left(1 + \dfrac{k^2}{n}\right)$. Hence for complete functions $\left(\epsilon = \dfrac{\pi}{2}\right)$ we have

$$\Pi(n) + \Pi\left(\frac{k^2}{n}\right) = \frac{\pi}{2\sqrt{\alpha}} + \mathrm{K}.$$

But here

$$\alpha = \left(1 - \frac{k^2}{\cos^2\theta'}\right)\sin^2\theta' = \left(\frac{a-x}{a+x}\right)^2 . \frac{z^2}{\rho^2}\,;$$

so that

$$\Pi\left(-\frac{k^2}{\cos^2\theta'}\right) = \frac{\pi}{2}\,\frac{\rho}{z}\,\frac{a+x}{a-x} + \mathrm{K} - \Pi(-\cos^2\theta'), \quad . \ . \ (6)$$

and (5) becomes

$$\frac{V}{m}=-\pi z+2\rho E+2\rho\frac{a-x}{a+x}\cdot K-2\frac{z^2}{\rho}\frac{a-x}{a+x}\Pi(-\cos^2\theta', k). \quad (7)$$

This expression holds without ambiguity for all positions of the point P, and it shows that for all points

on the axis, OV, of the plate, . $\frac{V}{m}=2\pi(\rho-z)$, . . (8)

on the perpendicular through B, $\frac{V}{m}=2\rho E-\pi z$, . . (9)

in the plane, between O and B, . $\frac{V}{m}=2\rho E+2\rho' K$, . (10)

in the plane, beyond B, . . . $\frac{V}{m}=2\rho E-2\rho' K$; . (11)

so that the points, occupying any of these positions, at which V has any assigned value can be easily found. Thus, to find the point on OV at which $\frac{V}{m}$ has the value C, we have for this point

$$\rho=z+\frac{C}{2\pi}.$$

Hence draw below AB, parallel to it and at the distance $\frac{C}{2\pi}$, a right line, meeting VO produced in O′; then the perpendicular to AO′ at its middle point meets OV in the required point.

Now every complete elliptic integral of the third kind can be expressed in terms of complete and incomplete functions of the first and second kinds. Thus, for a complete function with the parameter $-m$ it is known that, if we put $m=-k'^2\sin^2\epsilon$,

$$\Pi(-m,k)=K+\frac{\Delta'_\epsilon}{k'^2\sin\epsilon\cos\epsilon}\left\{\frac{\pi}{2}+KK'_\epsilon-KE'_\epsilon-EK'\right\}, \quad (12)$$

where Δ'_ϵ stands for $\sqrt{1-k'^2\sin^2\epsilon}$, and K'_ϵ, E'_ϵ stand for incomplete functions of the first and second kinds with modulus k' and amplitude ϵ.

In the present case,

$$m = \cos^2\theta', \quad k' = \frac{\rho'}{\rho} = \frac{\sin\theta'}{\sin\theta};$$

$\therefore\ \epsilon = \theta$, and

$$\Pi(-\cos^2\theta', k) = \mathrm{K} + \frac{\cos\theta'}{k'^2 \sin\theta\cos\theta}\left\{\frac{\pi}{2} + \mathrm{KK}'_\theta - \mathrm{KE}'_\theta - \mathrm{EK}'_\theta\right\}; \quad (13)$$

so that (7) becomes

$$\frac{\mathrm{V}}{2m} = z\{\mathrm{KE}'_\theta + \mathrm{EK}'_\theta - \mathrm{KK}'_\theta - \pi\} + \rho\mathrm{E} + \rho'\cos\theta\cos\theta' . \mathrm{K}. \quad (14)$$

It is evident that we may define the position of any point, P, in the plane of the figure by means of the two coordinates k and θ. Thus we have

$$\rho = \frac{2a}{\Delta'_\theta + k'\cos\theta},$$

$$\cos\theta' = \Delta'_\theta,$$

$$z = k'\rho\sin\theta, \quad \rho' = k'\rho.$$

Hence

$$\frac{\mathrm{V}}{4m} = \frac{a}{\Delta'_\theta + k'\cos\theta}\{\mathrm{E} + k'\sin\theta\,(\mathrm{KE}'_\theta + \mathrm{EK}'_\theta - \mathrm{KK}'_\theta - \pi)$$
$$+ k'\cos\theta . \mathrm{K}\Delta'_\theta\}. \quad . \ (15)$$

This, then, is the expression for the potential at any point in terms of the coordinates (k, θ) of the point. In particular, it gives the value (9) for any point on the perpendicular through B to the plate, since for such a point $\theta = \frac{\pi}{2}$, and then the coefficient of $k'\sin\theta$ within the brackets is equal to $-\frac{\pi}{2}$, by Legendre's well-known relation between the complete complementary functions, viz.,

$$\mathrm{KE}' + \mathrm{EK}' - \mathrm{KK}' = \frac{\pi}{2},$$

whatever the modulus k may be.

From (14) we can derive an expression for the conical angle subtended at any point, P, in space by a circle, *i. e.*, for the magnetic potential due to a current coinciding with the circle. It is well known that this conical angle is numerically equal to the component of the attraction, perpendicular to the plate, at P due to a uniform circular plate coinciding with the aperture of the circle—a result which is evident from the principle that the current can be replaced by a magnetic shell, or thin plate, the upper and lower surfaces of which are, of course, of opposite signs. But the resultant potential of these two indefinitely close plates is the difference between the value of V in (14) and the value which (14) assumes when $z+\Delta z$ is substituted for z; that is, the magnetic potential at P due to the current is $-\frac{dV}{dz}\,.\,\Delta z$, and the strength of the magnetic shell is $m\,.\,\Delta z$, which is i, the current in the circle; so that the magnetic potential is i multiplied by minus the differential coefficient of the right-hand side of (14) with respect to z.

Denote the function $\pi+KK'-KE'_\theta-EK'_\theta$ by the symbol Λ_θ, and for simplicity in the differentiation with respect to z (x being constant) write (14) in the form

$$\frac{V}{2m}=-z\Lambda_\theta+\rho E+(a^2-x^2)\frac{K}{\rho}. \quad . \quad . \quad . \quad (16)$$

Now

$$\left.\begin{aligned}\frac{dk}{dz}&=-\frac{kk'}{\rho}\sin\theta;\quad \frac{dk'}{dz}=\frac{k^2}{\rho}\sin\theta;\\ \frac{d\rho}{dz}&=k\sin\theta;\quad \frac{d\theta}{dz}=\frac{1}{k'\rho}\cos\theta;\end{aligned}\right\} \quad . \quad . \quad (17)$$

and, regarding P as determined by the coordinates (z, x) instead of (k, θ), we have

$$\frac{d}{dz}=\frac{dk}{dz}\cdot\frac{d}{dk}+\frac{d\theta}{dz}\cdot\frac{d}{d\theta};$$

but

$$\frac{d\Lambda}{dk}=\frac{K-E}{k}\,\frac{\sin\theta\cos\theta}{\Delta'_\theta},$$

$$\frac{d\Lambda}{d\theta}=\frac{Kk'^2\sin^2\theta-E}{\Delta'_\theta};$$

therefore

$$\frac{d\Lambda}{dz}=-\frac{1}{k'\rho}E\Delta'_\theta\cos\theta,$$

and we find $\frac{d}{dz}$ of the right-hand side of (16) equal to $-\Lambda_\theta+k'K\sin\theta$; so that if Ω is the conical angle subtended at P by the circle, or the magnetic potential per unit current in the circle, we have the very simple expression

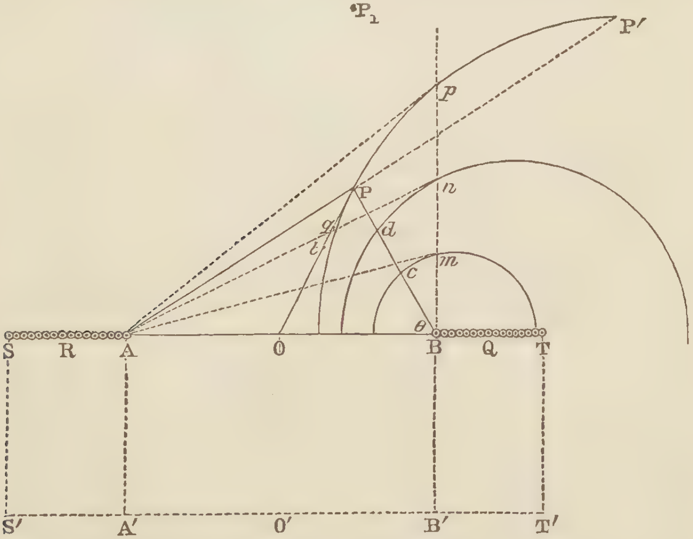
$$\Omega = 2\Lambda_\theta - 2k'\mathrm{K}\sin\theta. \quad . \quad . \quad . \quad . \quad (18)$$

Again, supposing that the depth, OO′ (fig. 1), of a coil consisting of a single series of circular currents is small compared with the distance of the point P from any part of it, the two terminal plates, ACB, A′C′B′ may be considered as close together, and the potential of the coil at P is the value of V in (16) minus the value obtained by putting $z+h$ for z, where $h=$OO′. Hence the potential in such a case is $-mh\Omega$, i. e., *at any point in space whose distance from every part of a coil is great compared with the depth of the coil, the potential is*

$$2mh(\Lambda_\theta - k'\mathrm{K}\sin\theta). \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The modulus k which appears in these equations, being $\left(1-\frac{\rho'^2}{\rho^2}\right)^{\frac{1}{2}}$, is constant at all points for which $\frac{\rho'}{\rho}$ is constant; *i. e.*, at all points on any circle which cuts that described on AB as diameter orthogonally. The circles which cut this latter orthogonally, having their centres on AB, are most readily drawn by joining A to points, m, n, p, . . . (fig. 3) on the perpendicular at B to AB, and drawing perpendiculars at

Fig. 3.



$m, n, p, \ldots$ to Am, An, Ap, ... ; the points of intersection of AB produced with these perpendiculars are the centres of the circles. If C is the centre of the orthogonal circle through m, we know that the constant, $1-\frac{\rho'^2}{\rho^2}$, on this circle is $\frac{\text{AB}}{\text{AC}}$, *i. e.*, $\cos^2 m\text{AB}$. Hence if $m\text{AB}=\beta$, we have

$$k=\cos\beta; \qquad k'=\sin\beta. \quad . \quad . \quad . \quad . \quad (20)$$

The field due to the plate AB is most readily mapped out by describing a large number of very close circles of the orthogonal system for a regular gradation of the values mAB, nAB, pAB, . . . of β, drawing a line BP in the assigned direction θ, and from Legendre's tables of Elliptic Integrals taking out the values of K, E, K'_θ, E_θ.

The properties of the orthogonal circles lead to some simple results with regard to potentials. Thus, if any line, AP, is drawn from A cutting any circle of the series in P and P′, the lines joining P and P′ to B are equally inclined to AB, *i. e.*, $\angle\text{ABP}'=\pi-\theta$.

Now if in Λ_θ we put $\pi-\theta$ for θ, we have, in virtue of Legendre's relation between complete complementary integrals, $\Lambda_{\pi-\theta}=\pi-\Lambda_\theta$, *i. e.*,

$$\Lambda_{\pi-\theta}+\Lambda_\theta=\pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Hence, from (18), if Ω, Ω' are the conical angles subtended at P, P′ respectively by the circle (or plate) AB, we have the remarkable relation

$$\Omega+\Omega'=2\pi-4k'\text{K}\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Again, if V, V′ are the potentials at P, P′ due to the plate, we have from (15)

$$\frac{\text{V}}{\text{AP}}+\frac{\text{V}'}{\text{AP}'}=2m(2\text{E}-\pi k'\sin\theta), \quad . \quad . \quad . \quad (23)$$

a result which enables us to lay down the field at all points to the right of the perpendicular Bp when the field to the left of Bp is known.

Supposing now that instead of a single wire of diameter AB, we have a series of wires forming a coil contained between the diameter AB and the diameter ST, *i. e.*, the breadth of the coil is BT or AS ; then in calculating the potential at

P we shall have to find the potentials due to a series of circular plates, each of surface-density m, and to add these potentials together. But observe that the potential at P due to any plate, AB, of radius a is of the form

$$a \cdot \phi(k, \theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where $\phi(k, \theta)$ is the coefficient of a in (15), and $\phi(k, \theta)$ is a function of θ and θ', the angles PAB and PBA. Hence if we take a plate of radius OQ, and from B draw Bq parallel to QP and meeting OP in q, the potential of this plate at P is to the potential of the plate AB at q as OQ is to OA; for, if AR=BQ, the angles qBA and qAB are equal, respectively, to PQR and PRQ. Hence, if r=OQ and V_q is the potential at q due to the plate AB (of radius a), the resultant potential at P due to the series of plates of radii extending from OB to OT is

$$\frac{1}{a}\Sigma r \cdot V_q, \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

the points q on OP ranging from t to P, where Bt is parallel to PT.

Of course any plate of the series may be taken instead of AB as the reference plate.

Thus, *the resultant potential, due to all the plates, is calculated from values of the potential of any one plate at a series of points ranged along the radius vector* OP.

Pass now to the consideration of the practical problem in hand, viz., the potential at P due to a coil of depth OO′, *i. e.*, we have to consider the whole of the spaces BTT′B′ and ASS′A′ filled with wire traversed by a current of strength i. We have already seen that we have to subtract from the potential at P due to a series of uniform attracting plates, each of surface-density m, ranging from the radius OB to the radius OT, the potential at P due to the lower series, each of surface-density m, and ranging from radius OB′ to radius OT′. It merely remains to express m in terms of current-density. If C is the total quantity of current traversing (at right angles to the plane of the paper) a unit area (square centimetre) of the space BTT′B′, the quantity flowing in a filament of depth dy and breadth dr is $Cdydr$. Now this filament is replaced by the magnetic shell of radius r and thickness dy;

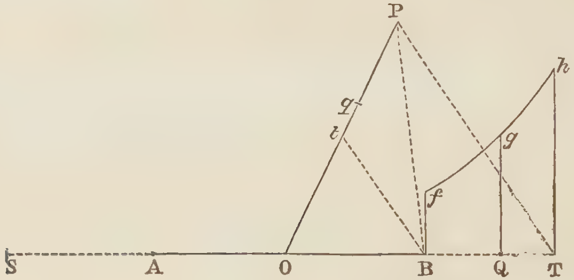
and since we know that the strength of the shell is equal to the current in the filament, we have $m \, . \, dy = Cdydr, \therefore m = Cdr$; hence (25) becomes, from (15),

$$4C\int r \, . \, \phi(k, \theta) dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

which is the potential due to the upper series of plates, OB, . . . OT.

This quantity may be graphically represented and calculated as follows. Let a very close series of curves representing a series of constant values of the function $\phi(k, \theta)$ for the plate AB be drawn; draw OP, and at each point, T, Q, . . . B, of the breadth BT of the coil draw an ordinate, Th, Qg, . . . Bf (fig. 4), equal to the product of r and the value of $\phi(k, \theta)$ at the corresponding point, t, q, . . . P, of the line OP: these ordinates will form by their extremities a curve, $hg \ldots f$, the area of which multiplied by four times the current-density in the space occupied by the coil is the potential at P due to the upper series of plates, OB, OQ, . . . OT.

Fig. 4.



Now if we take the point P_1 such that PP_1 is equal and parallel to OO′, the depth of the coil, the potential at P due to the lower series of plates, OB′, . . . OT′, is equal to that at P_1 due to the upper series. Hence, if A and A_1 are the areas of the curve $hg \ldots f$ and the corresponding curve for the point P_1, the total potential at P due to the complete coil is

$$4C(A - A_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The curve *hgf* passes, of course, through the point O ; and when the line OP coincides with the axis, OO′, of the coil, the curve is an hyperbola ; for, in this case

$$\phi(k, \theta) = \frac{\pi}{2} \cdot \frac{1 - \sin\theta}{\cos\theta},$$

and $r = \mathrm{OP} \cot\theta$.

We may, if we please, express r in terms of (k, θ), and draw the curve *hgf* by a different rule. Thus,

$$r = \mathrm{OP} \,.\, \frac{\Delta'_\theta + k' \cos\theta}{\sqrt{(\Delta'_\theta - k' \cos\theta)^2 + 4k'^2 \sin^2\theta}},$$

and we can make the ordinate of the curve equal to

$$\mathrm{OP} \,.\, \frac{\mathrm{E} - k' \sin\theta \,.\, \Lambda_\theta + k'\mathrm{K} \cos\theta \,.\, \Delta'_\theta}{\sqrt{(\Delta'_\theta - k' \cos\theta)^2 + 4k'^2 \sin^2\theta}}.$$

Discussion.

Prof. Perry hoped the work Prof. Minchin had done so well for circles and cylinders would be extended to cylindrical coils of rectangular cross-section. It was most important to be able to find the shape of the field produced by such coils.

Prof. S. P. Thompson enquired if there was any way of deducing the expression for the magnetic force at a point other than that given in the paper on the " Magnetic Field of a Circular Current " (*antea*, p. 204).

In reply, Prof. Minchin explained how the formula followed at once from the fundamental theorem that magnetic force is the cure of the vector potential. This was based on Laplace's expression for the force between a magnetic pole and an element of current, which had been proved experimentally.

XXXV. *A Potentiometer for Alternating Currents.*

By JAMES SWINBURNE[*].

ARGUMENT.

A direct pressure is balanced against an alternating pressure by means of a differential electrometer with single-fibre suspension, a null method being employed. An alternating and a direct current are similarly compared by means of a differential dynamometer with the controlling spring removed.

ONE of the chief difficulties in the way of the accurate measurement of alternating pressures and currents arises in the calibration of the instruments. Some of the alternating-current instruments, such as those on the electro-dynamometer principle, give the same reading whether a direct or alternating current is used; and electrometers and some forms of electromagnetic instruments with soft iron cores are also adapted equally well for direct and alternating-current work. This is not the case, however, with a large class of instruments, and the only method at present available for calibrating such instruments is comparing them with another instrument which has been calibrated by means of a direct current. Such a practice allows errors to creep in. No careful electrician would trust to direct-current instruments calibrated second-hand, and accuracy is just as important in alternating-current work. In addition to this, the collection of cells and resistances which are now included under the name "potentiometer" is exceedingly useful in any laboratory where it can be permanently set up, as it measures currents and pressures throughout an enormous range with an accuracy that is quite unattainable by means of any other forms of voltmeter or ampere-meter. It is therefore of the highest scientific importance to be able to extend the potentiometer method into alternating-current work, so that comparisons can be made directly with a standard cell.

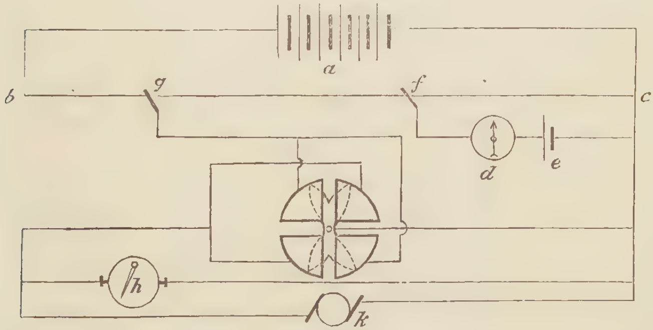
In October 1891[†] I described two forms of alternating-current so-called ohmmeter. These instruments measured either a quantity R such that E^2/R was the power when E was the effective or virtual pressure, or a quantity r such that Cr^2 was the power when C was the virtual or effective current. Of course in a circuit with capacity or self-induction, R and r are not equal. One form of alternating ohmmeter was electrostatic, the other electromagnetic. The electrostatic ohmmeter can be coupled up in such a way as to compare two pressures, and one of these may be alternating and the other direct. The electrostatic instrument has the advantage over the electromagnetic in requiring only one connexion to the moving system, or needle, and that connexion has not to carry any appreciable current. The needle may therefore be suspended by means of a single silk fibre, connexion being made through a hanger dipping into water, and this hanger may end in a vane to make the instrument dead-beat. As the ohmmeter does not need any torsional control, but should be, on the contrary, as free as possible, this form of instrument can be made exceedingly sensitive. For potentiometer work it is best not to use the instrument for giving the ratio of two electromotive forces, but to design it as a differential galvanometer which shows whether the pressures are equal or not.

---

[*] Read December 8, 1893.

[†] 'Industries,' October 30, 1891.

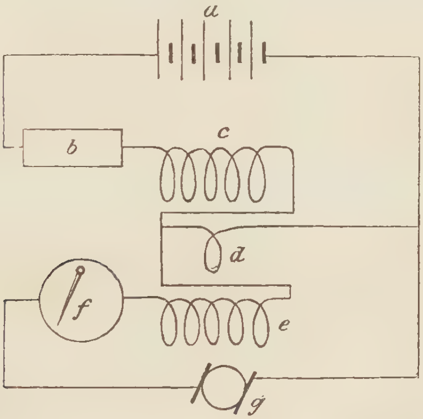
Fig. 1.



The disposition of the whole apparatus is shown diagrammatically in fig. 1. In this scheme a is a battery maintaining a current in the resistance bc, which is regulated by an adjustable resistance so that the standard cell e is balanced when its contact f is touched down at the mark corresponding to its electromotive force at the temperature at which it stands; *k* is an alternating dynamo, and *h* is the alternating voltmeter to be calibrated. The differential electrometer is connected up as shown. The double fishtail-shaped needle is pulled one way by a force varying as the square of the direct electromotive force, and the other way by a force varying as the square of the alternating pressure. The fishtail shape is necessary to ensure that the needle is in stable equilibrium when the forces are equal. Fig. 1 is, of course, a mere diagram. Resistance-coils for adjustment are left out, and *b c* is shown as a stretched wire, whereas it is made up of resistance-coils. As to sensitiveness, it depends on the construction of the electrometer; but, as already explained, there is no controlling force except that due to the suspending fibre. Take as an example an instrument that will indicate one volt when there is no pressure on the other side: it will admit of the comparison of two pressures of approximately 100 volts within one in ten thousand; and it will serve for comparing 2000 volts within one in four million. To find the true zero of the instrument the four quadrants are connected together. The key for making this connexion is not shown.

One of the great advantages of the potentiometer method in direct-current work is the ease with which large currents can be measured by the fall of potential over very small resistances. As the electrometer must be used idiostatically in alternating work it is not sensitive to very small pressures; and the method given is, unfortunately, useful only for pressure measurements. Accurate current measurements can be taken by the use of a differential dynamometer. This instrument has two fixed and one moving coil. The direct circuit is through one fixed and the alternating through the other fixed coil, and both circuits are led through the moving coil. The controlling spring is removed. This method is somewhat more complicated than in the case of pressure, and mercury contacts are necessary. The arrangement is shown in fig. 2. The battery, *a*, supplies a direct current, which is measured by the fall of potential over the low resistance, *b*, and which passes in one of the fixed coils, *c*, and the moving coil, *d*. The alternating-current circuit is led through the circuit, *f*, whose current is to be measured, and through the other fixed coil, *e*, and the moving coil, *d*. There is a slight error due to the whole of the alternating circuit being in shunt to the moving coil considered as part of the direct-current circuit, and *vice versâ*; but this is very minute. If *c* and *e* are wound close together so that there is no appreciable time-lag in any current induced in *c* by *e*, the small error due to the mutual induction of these coils cancels out.

Fig. 2.



Of course a double dynamometer might be used, but then four mercury connexions are needed. It is not unlikely that a method with no mercury contacts can be made available, and greater accuracy could then be attained.

Discussion.

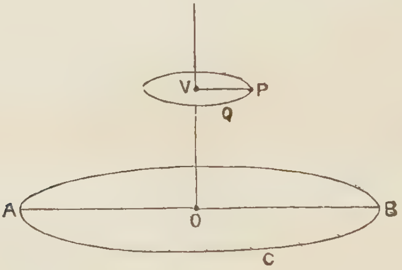
Prof. S. P. Thompson inquired if the fishtail-shaped needle of the electrometer was novel.

Mr. Blakesley said the author had mentioned the needle previously. He (Mr. Blakesley) thought the name "potentiometer" was not very suitable. In effect the so-called measurement of *pressures* was a comparison of two *powers*.

XXXVI. *Calculation of the Coefficient of Self-Induction of a Circular Current of given Aperture and Cross-Section.* By Professor G. M. Minchin, *M.A.*[*]

Let ACB (fig. 1) represent a circular wire in which a current of strength i is circulating; let O be its centre and OV its axis (perpendicular to its plane); let P be any point in space and through P describe a circle, PQ, parallel to the plane of the current, its centre being V on the axis.

Fig. 1.



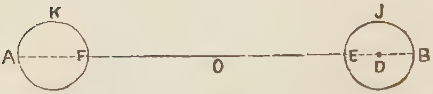


Fig. 2.

It is required to calculate the normal flux of magnetic force passing through the circle PQ. If $VP = x$, and the vector potential of the current at P is G (this latter being, of course, perpendicular to VP and parallel to the plane of the current), the component, Z, of the magnetic force at P parallel to OV is given by the expression

$$Z = \frac{dG}{dx} + \frac{G}{x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(See my paper on the "Magnetic Field of a Circular Current," *antea*, p. 204). This can be written

$$Z = \frac{1}{x}\,\frac{d(Gx)}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

---

[*] Read December 8, 1893.

The function $G.x$ is the same as Stokes's *current function* which exists for fluid motion which is symmetrical about an axis. (See Basset's 'Hydrodynamics,' vol. i. p. 12.)

Taking a circular strip of radius x and breadth dx at P, the flux of force through the strip is $2\pi Z x\,dx$, i. e.,

$$2\pi \frac{d(Gx)}{dx} \,.\, dx.$$

Hence, integrating this from $x=0$ to $x=\mathrm{VP}$, we find that the total normal flux through the circle PQ is the value of

$$2\pi G x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

at P.

Let fig. 2 represent the cross-sections of the wire at A and B in fig. 1 made by a plane through the axis OV, the radius of each being c, while the radius, OD, of the central filament of the wire is a (as in my paper on the "Magnetic Field close to the Surface of a Wire conveying an Electrical Current," *antea*, p. 379).

Then we shall calculate *the total normal flux of force through any surface which is intersected once in the positive direction by every tube of force emanating from the given current.*

This quantity, divided by the current-strength, is the coefficient of self-induction of the current. Taking the general case, viz., that in which the current-density at every point in the cross-section of the wire varies inversely as the distance of this point from the axis OV—we have found (*antea*, p. 398) that at any point, P, close to the wire

$$\begin{aligned} Gx = 4ai \Bigg\{ \frac{L}{2} - 1 - \frac{m}{4a} \cos\phi \left[L - 1 - \frac{c^2}{4m^2} \right] \\ + \frac{1}{16a^2} \Bigg[L\left(\frac{3c^2}{4} - \frac{m^2}{2} \right) + \frac{m^2}{2} - \frac{15c^2}{4} \\ - \cos 2\phi \left\{ L \frac{m^2}{2} + \frac{c^2}{4} - m^2 - \frac{c^2}{8m^2} \right\} \Bigg] \Bigg\} \quad . \quad (4) \end{aligned}$$

where $m = \mathrm{PD}$, $\phi = \angle \mathrm{PDA}$, $L = \log_e \dfrac{8a}{m}$.

The surface through which we shall take the flux of force is that which is represented in section by BJEOFKA, *i.e.*, a surface consisting of the upper half portion of the anchor-ring formed by the wire and of its central aperture (which latter is a circle whose diameter is FE). Obviously this surface is intersected by all the tubes of force. Any surface starting from B and going round to A, *i. e.*, any surface having the circle of diameter AB for bounding edge, would do equally well, so far as the above condition is concerned ; but the calculation is simpler for the first.

The flux through the aperture FE is, then, the value of $2\pi G \, . \, x$ at E, which is obtained by putting $m=c$, $\phi=0$ in the above.

Thus the flux through the aperture is

$$8\pi ai\left\{\frac{L}{2}-1-\frac{c}{4a}\left(L-\frac{5}{4}\right)-\frac{c^2}{128a^2}(2L+19)\right\}. \quad . \ (5)$$

To calculate the normal flux through the upper half of the anchor-ring, we must take the value of the magnetic potential, Ω, at any point close to the ring. This is the resultant conical angle subtended by the circuit at the point, multiplied by i; and it is therefore (*antea*, p. 392) given by the equation

$$\Omega=i\left\{2(\pi-\phi)-\frac{\sin\phi}{a}\left(mL+\frac{c^2}{4m}\right)+\frac{1}{16a^2}\left[8c^2(\pi-\phi)\right.\right.$$
$$\left.\left.-\left\{(6L-5)m^2+c^2+\frac{c^4}{2m^2}\right\}\sin 2\phi\right]\right\}. \quad . \ (6)$$

Now the normal force at any point on the anchor-ring is the value of $-\frac{d\Omega}{dm}$ with $m=c$; it is therefore

$$i\left\{\left(L-\frac{5}{4}\right)\frac{\sin\phi}{a}+\frac{c}{16a^2}(12L-17)\sin 2\phi\right\}. \quad . \ (7)$$

If P is any point on the anchor-ring, the distance of P from OV is $a-c\cos\phi$, and the area of a narrow circular strip of the ring parallel to the plane of the aperture of the ring is $2\pi(a-c\cos\phi)\, . \, cd\phi$; and this multiplied by (7) is the normal flux of force through the strip. Integrating the product from $\phi=0$ to $\phi=\pi$, we get the flux through the whole of the upper half of the ring. The result is simply

$$\frac{2\pi ic}{a}\left(L-\frac{5}{4}\right)\int_0^\pi (a-c\cos\phi)\sin\phi \, d\phi,$$

the second term in (7) being neglected because it gives a term of the third order. Thus this part of the flux is

$$4\pi ci\left(\mathrm{L}-\frac{5}{4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Adding this to (5), we have the whole flux sought, divided by i, equal to

$$\pi\left\{4a(\mathrm{L}-2)+2c\left(\mathrm{L}-\frac{5}{4}\right)-\frac{c^2}{16a}(2\mathrm{L}+19)\right\}, \quad . \quad (9)$$

where $\mathrm{L}=\log_e\dfrac{8a}{c}$. This, then, is the Coefficient of Self-Induction. If in this expression a and c are taken in centimetres, the result is the coefficient measured in absolute units; and if this is divided by 10^9, we have the coefficient of self-induction in secohms.

Thus, for example, a circular current running in a wire the diameter of whose cross-section is 2 millim., while the diameter of its central filament is 2 centim., has a coefficient of self-induction of about 59·207 absolute units, or $\dfrac{59{\cdot}207}{10^9}$ secohms; and if the dimensions of the cross-section were neglected, this number would be 58·866.

Clerk Maxwell (Elec. and Mag. vol. ii. art. 704) gives the coefficient of self-induction as $4\pi a(\mathrm{L}-2)$, which agrees with (9) in the principal term.

In the same way we may find the coefficient for a superficial current in the wire. For (*antea*, p. 221) if q is the total quantity of the superficial current, we have the value of Gx at E equal to

$$2q\left[a(\mathrm{L}-2)-\frac{c}{4}(2\mathrm{L}-1)+\frac{c^2}{16a}\left(\mathrm{L}+\frac{11}{4}\right)\right],$$

while from the value of the potential we find the normal flux of force through the upper half of the anchor-ring equal to

$$2\pi qc(2\mathrm{L}+1).$$

Hence the Coefficient of Self-Induction is

$$\pi\left\{4a(\mathrm{L}-2)+2c\left(\mathrm{L}+\frac{3}{2}\right)+\frac{c^2}{16a}(4\mathrm{L}+11)\right\}, \quad . \quad (10)$$

which is somewhat greater than the value (9) for a steady current.

XXXVII. *Note on a New Photometric Method, and a Photometer for same. By* Mr. J. B. Spurge.* (Abstract.)

The method consists in using two diffusing screens (illuminated respectively by the lights to be compared) as secondary sources, and adjusting to equality the luminosity of equidistant internal surfaces by varying the apertures through which the light passes from the screens to these surfaces. By reducing the sizes of the apertures, the author has been enabled to compare lights of different colour; for when of sufficiently feeble intensity, coloured lights are indistinguishable from white or grey.

The photometer is made up of two tubes mounted at 45° to an axis, about which one of them is capable of rotating. When in the same horizontal plane, the axes of the tubes form the sides of an isosceles right-angled triangle, at the middle of whose hypothenuse the light to be tested is placed; this illuminates one of the screens, whilst the standard light shines on the other. These screens, used as secondary sources, are situated a short distance away from the outer ends of the tubes, whilst the inner surfaces of the near ends of the tubes are viewed by means of a mirror. By turning the movable tube about the inclined axis, and rotating the source about a vertical axis, the illuminating effect of the source in any direction can be tested.

Discussion.

Capt. Abney said the law of inverse squares was not true for weak lights, for the proportions in which the light from sources of equal intensity had to be reduced to appear white or grey depended greatly on the colour, being much greater for violet than for red. Only for the yellow-green rays was the ordinary law of illumination true when the intensities were feeble.

Mr. Blakesley, Prof. S. P. Thompson, and the President also took part in the discussion.

---

* Read January 26, 1894.

INDEX.

A.

Page

Acquisition of conducting-power by a series of discrete metallic particles, on the sudden 461
Action of electromagnetic radiations on films containing metallic powders 455
Alternating currents, a potentiometer for 514
Applicability of Lagrange's equations of motion in a general class of problems; with especial reference to the motion of a perforated solid in a liquid 256

B.

Baily, Walter, on the construction of a colour map 1
Baly, E. C. C., on the separation and striation of rarefied gases under the influence of the electric discharge 147
Barrell, F. R., Thomas, G. L., and Young, S., on the separation of three liquids by fractional distillation 422
Blakesley, T. H., on the differential equation of electrical flow 217
Boiling-points of different liquids at equal pressures 142
Boltzmann, Ludwig, on the methods of theoretical physics 336
Boys, C. V., on the drawing of curves by their curvature 345
Bryan, G. H., on a hydrodynamical proof of the equations of motion of a perforated solid, with applications to the motion of a fine rigid framework in circulating liquid 186
Burton, C. V., on plane and spherical sound-waves of finite amplitude 161
——, on the applicability of Lagrange's equations of motion in a general class of problems; with especial reference to the motion of a perforated solid in a liquid 256

C.

Calculation of the coefficient of self-induction of a circular current of given aperture and cross-section 518
—— of the magnetic field of a current running in a cylindrical coil 502
Circular current, the magnetic field of a 204
Colour map, on the construction of a 1
Concentric spherical shells, on the magnetic shielding of 462
"Corresponding" temperatures, pressures, and volumes, note on the generalizations of Van der Waals regarding 447

Page
Critical density, on the determination of the 134
—— volume, on the determination of the 137
Croft, W. B., two lecture-room experiments 421

D.

Determination of the critical density 134
—— of the critical volume 137
Determinations of molecular refraction and dispersion, notes on some recent 153
Differential equation of electrical flow, on the 217
Diffusion of light, on the 10
Discrete metallic particles, on the sudden acquisition of conducting-power by a series of 461
Drawing of curves by their curvature 345
Dynamics, on the foundations of 289

E.

Electrical flow, on the differential equation of 217
Electric current, on the magnetic field close to the surface of a wire conveying an 379
—— discharge, separation and striation of rarefied gases under the influence of 147
Electrodeless vacuum-tubes, luminous discharges in 265
Electromagnetic radiations, the action of, on films containing metallic powders 455
Equations of motion, on the applicability of Lagrange's, in a general class of problems 256
—— of motion of a perforated solid, a hydrodynamical proof of 186
Everett, J. D., on a new and handy focometer 180
Experiments in electric and magnetic fields, constant and varying . 3

F.

Films containing metallic powders, on the action of electromagnetic radiations on 455
Focometer, on a new and handy 180
Form of influence-machine, on a new 403
Foundations of dynamics, on the 289
Fractional distillation, on the separation of three liquids by........ 422

G.

Generalizations of Van der Waals regarding "corresponding" temperatures, pressures, and volumes, note on the 447
Gladstone, J. H., notes on some recent determinations of molecular refraction and dispersion 153

H.

Page

Hydrodynamical proof of the equations of motion of a perforated solid, with applications to the motion of a fine rigid framework in circulating liquid 186

I.

Influence-machine, on a new form of 403
——, on an 406
Influence of the electric discharge, separation and striation of rarefied gases under the 147

L.

Lagrange's equations of motion in a general class of problems, on the applicability of 256
Laws of molecular force, on the 30
Lecture-room experiments, on two 421
Light, on the diffusion of 10
Liquid friction, on 236
Liquids at equal pressures, on the boiling-points of different 142
Lodge, Oliver, on the foundations of dynamics 289
——, on the sudden acquisition of conducting-power by a series of discrete metallic particles 461
Long-distance telephony 413
Luminous discharges in electrodeless vacuum-tubes 265

M.

Magnetic field of a circular current 204
—— —— of a current running in a cylindrical coil, calculation of the 502
—— —— close to the surface of a wire conveying an electric wire 379
—— shielding of concentric spherical shells 462
Methods of theoretical physics, on the 336
Minchin, G. M., calculation of the coefficient of self-induction of a circular current of given aperture and cross-section 518
——, calculation of the magnetic field of a current running in a cylindrical coil 502
——, on the action of electromagnetic radiations on films containing metallic powders 455
——, on the magnetic field of a circular current 204
——, on the magnetic field close to the surface of a wire conveying an electrical current 379
Molecular force, on the laws of 30
—— refraction and dispersion, notes on some recent determinations of 153
Myers, J. E., on a new volumenometer 372

N.

Page

New and handy focometer 180
—— form of influence-machine 403
—— photometer 354
—— photometric method, and a photometer for same 522
—— volumenometer 372
Note on the generalizations of Van der Waals regarding "corresponding" temperatures, pressures, and volumes 447
Notes on photometry 361
—— on the construction of a colour map 1
—— on some recent determinations of molecular refraction and dispersion 153

P.

Perry, John, on liquid friction 236
——, on long-distance telephony 413
Photometer, on a new 354
Photometric method, note on a new 522
Photometry, some notes on 361
Pidgeon, W. R., on an influence-machine 406
Plane and spherical sound-waves of finite amplitude 161
Potentiometer for alternating currents 514

R.

Rarefied gases under the influence of the electric discharge, separation and striation of 147
Recent determinations of molecular refraction and dispersion, notes on some 153
Rimington, E. C., on luminous discharges in electrodeless vacuum-tubes 265
—— and Smith, Wythe, on experiments in electric and magnetic fields, constant and varying 3
Rücker, A. W., on the magnetic shielding of concentric spherical shells 462

S.

Self-induction of a circular current of given aperture and cross-section, calculation of the coefficient of 518
Separation and striation of rarefied gases under the influence of the electric discharge 147
—— of three liquids by fractional distillation 422
Smith, Wythe, and Rimington, E. C., on experiments in electric and magnetic fields, constant and varying 3

Page
Spherical shells, on the magnetic shielding of concentric 462
—— sound-waves of finite amplitude, on plane and 161
Spurge, J. B., note on a new photometric method, and a photometer for same 522
Sudden acquisition of conducting-power by a series of discrete metallic particles, on the 461
Sumpner, W. E., on the diffusion of light 10
Surface of a wire conveying an electric current, on the magnetic field close to the 379
Sutherland, W., on the laws of molecular force 30
Swinburne, James, on a potentiometer for alternating currents 514

T.

Telephony, on long-distance 413
Theoretical physics, on the methods of 336
Thomas, G. L., and Young, S., on the determination of the critical density 134
——, Barrell, F. R., and Young, S., on the separation of three liquids by fractional distillation 422
Thompson, S. P., some notes on photometry 361
Trotter, A. P., on a new photometer 354

V.

Vacuum-tubes, luminous discharges in electrodeless 265
Volumenometer, on a new 372

W.

Wimshurst, James, on a new form of influence-machine 403
Wire conveying an electric current, on the magnetic field close to the surface of a 379

Y.

Young, Sydney, note on the generalizations of Van der Waals regarding "corresponding" temperatures, pressures, and volumes . 447
——, on the boiling-points of different liquids at equal pressures .. 142
——, on the determination of the critical volume 137
——, and Thomas, G. L., on the determination of the critical density 134
——, ——, and Barrell, F. R., on the separation of three liquids by fractional distillation 422

Printed by TAYLOR and FRANCIS, Red Lion Court, Fleet Street.

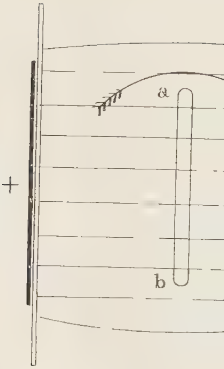
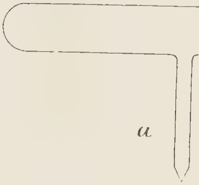
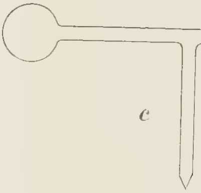


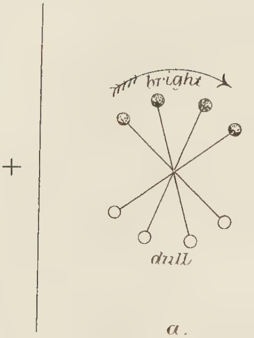
Fig. 1.



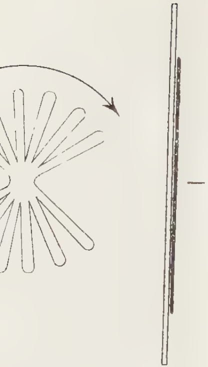
a



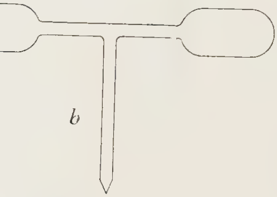
c

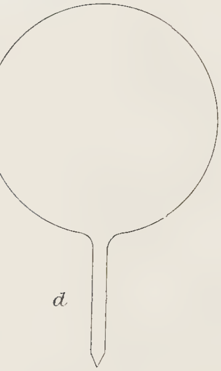


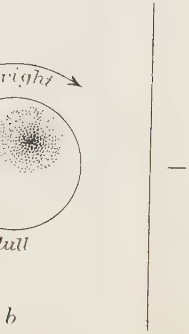
a.



g. 2.







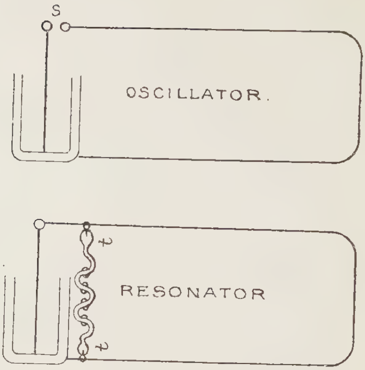


Fig. 5.

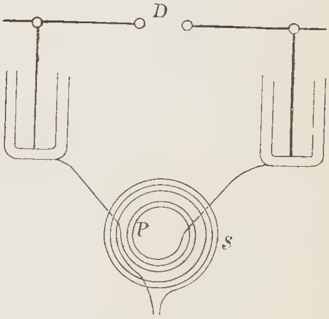
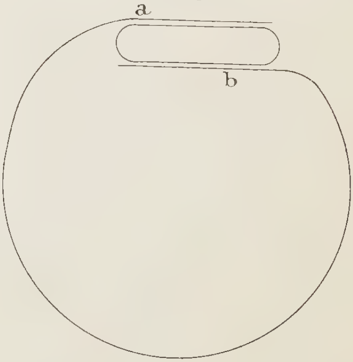


Fig. 7.

Fig. 6.



Mintern Bros. lith.

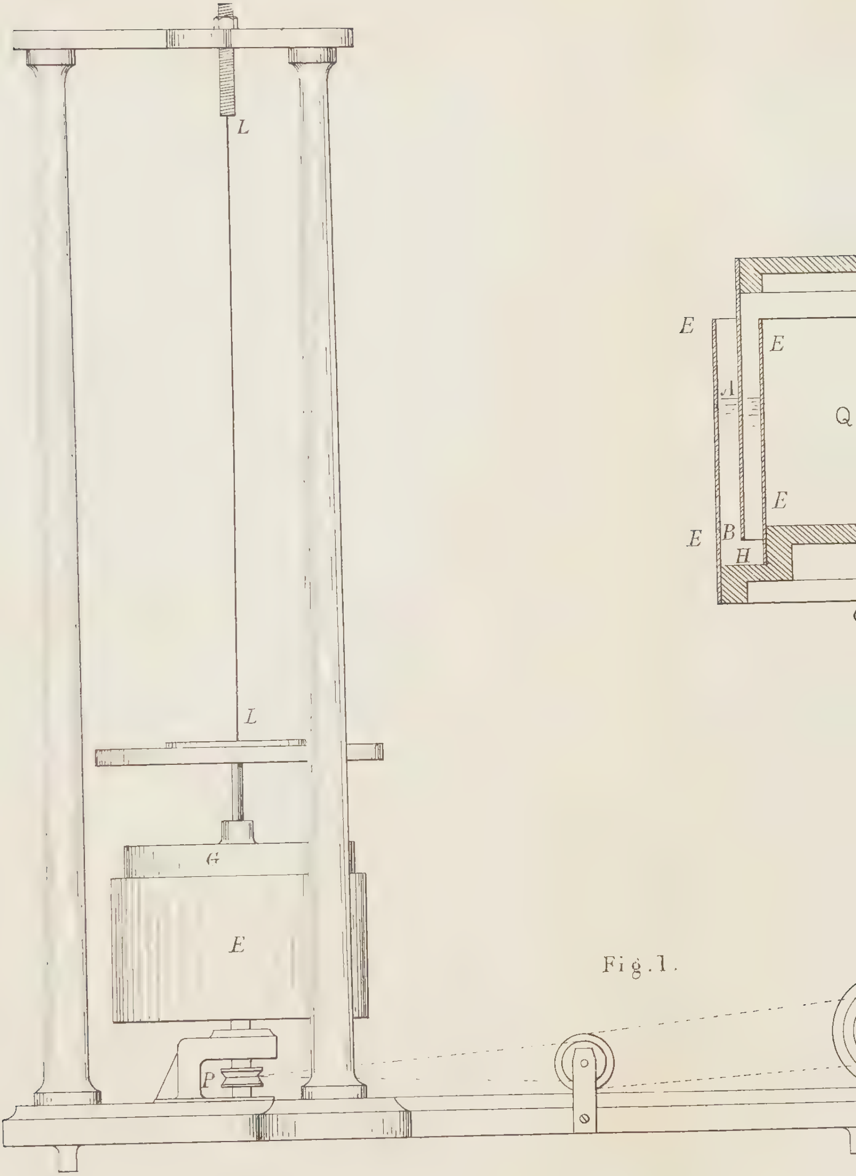


Fig. 1.

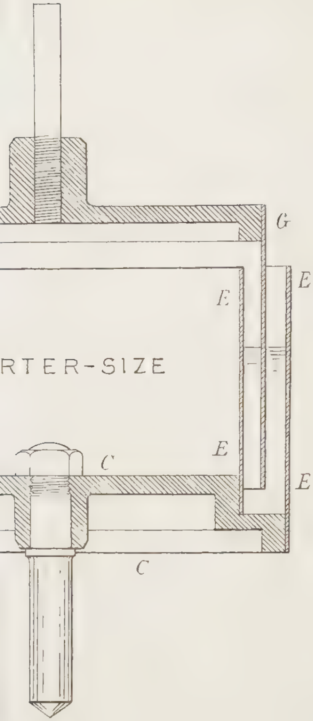
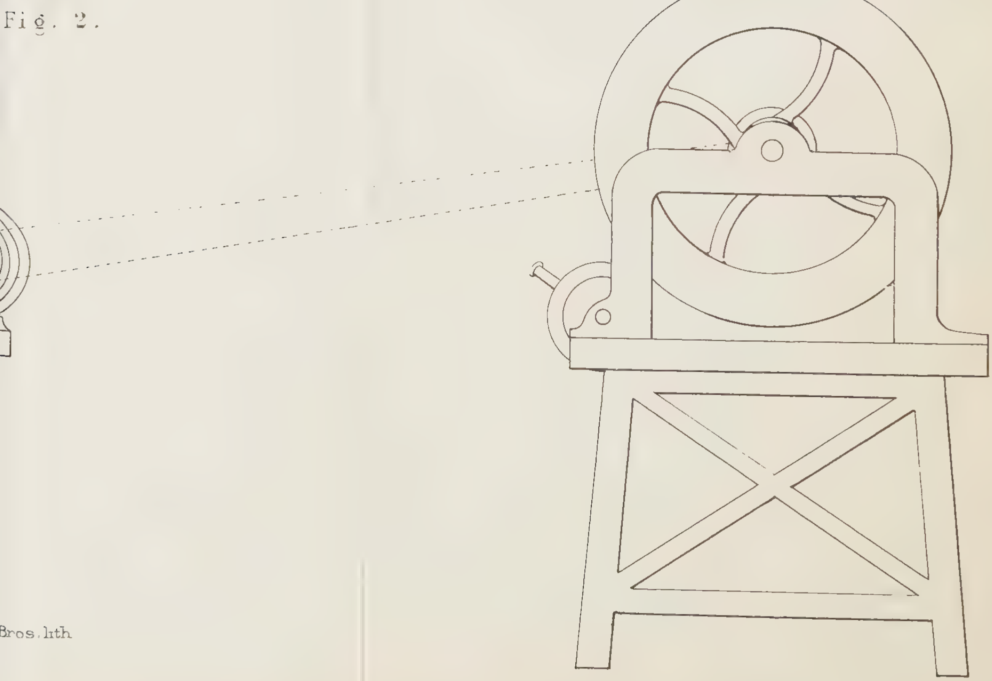
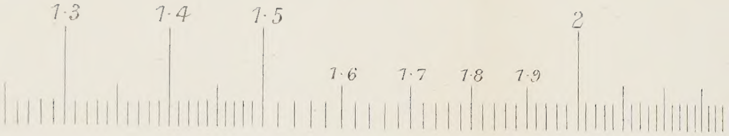


Fig. 2.



Bros. lith



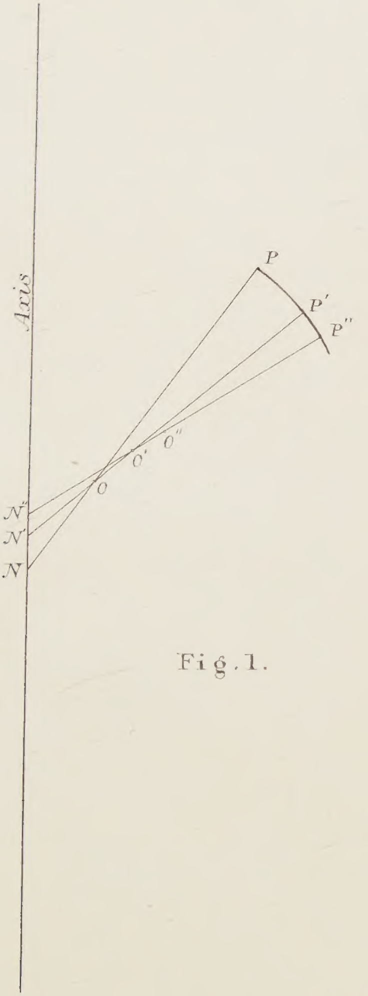
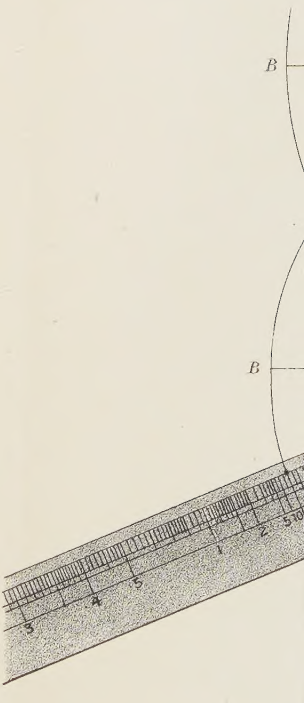
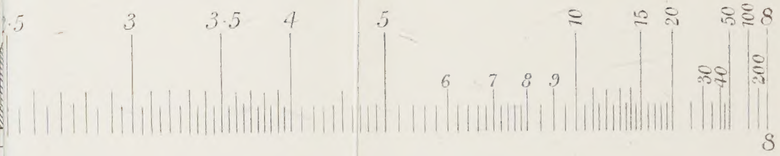


Fig.1.

F



Minter



. 3.

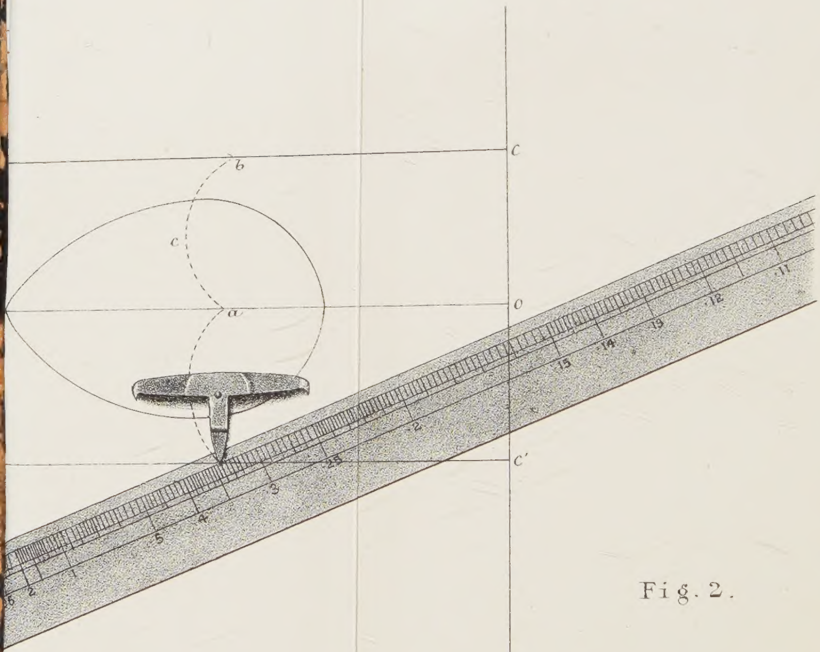


Fig. 2.

Bros. lith.